LOGARITHMIC SOBOLEV INEQUALITY FOR THE INHOMOGENEOUS ZERO RANGE PROCESS

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Abstract. We prove that the logarithmic Sobolev constant for the inhomogeneous symmetric nearest neighbour zero range process on a cube of size N^d grows as N^2. We apply this result to the inhomogeneous process which arises in the study of the homogeneous version of the zero range interacting particle system with colours.

1. Introduction

The logarithmic Sobolev inequality is a spectral bound which provides much information about decay to equilibrium of the dynamics of a stochastic process.

Consider a process governed by reversible dynamics described by a generator $\mathcal{L}$, with semi-group P_t and an invariant measure μ. The Dirichlet form is defined as $D_\mu(f) = \mu[f(-\mathcal{L})f]$. A logarithmic Sobolev inequality is a statement which says that the entropy, $H(f|\,\mu) = \mu[f \log f]$, is bounded by a constant times the Dirichlet form

$$H(f|\,\mu) \leq C_{\mathrm{LS}}\ D_\mu(\sqrt{f}), \tag{1.1}$$

for all densities f. Additionally, the logarithmic Sobolev inequality implies exponential decay of both the $L^2(\mu)$ norm and entropy of $P_t f$.

The Poincaré inequality is defined as the bound, uniform in f,

$$\mathrm{Var}_\mu[f] \leq C_{\mathrm{SG}}\ D_\mu(f), \tag{1.2}$$

where $\mathrm{Var}_\mu[f]$ is the variance of the function f with respect to the measure μ. The inequality states that the dynamics of the process have a spectral gap of order C_{SG}^{-1} and hence there is exponential decay to equilibrium in the L^2 sense. That is, we have that

$$\mathrm{Var}_\mu[P_t f] \leq e^{-2t/C_{\mathrm{SG}}}\mathrm{Var}_\mu[f],$$

where P_t is the semi-group of the process.

An intermediate spectral bound may be established via the following entropy dissipation inequality,

$$H(f|\,\mu) \leq C_{\mathrm{ED}}\ \mu[f(-\mathcal{L})\log f], \tag{1.3}$$

uniformly in positive functions f. Because $\partial_t H(P_t f|\,\mu) = \mu[P_t f(-\mathcal{L})\log P_t f]$, this implies that

$$H_\mu[P_t f] \leq e^{-2t/C_{\mathrm{ED}}} H_\mu[f],$$

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again for positive functions. One can also show that,

$$C_{\mathrm{SG}} \le 2C_{\mathrm{ED}} \le \frac{1}{2} C_{\mathrm{LS}}, \tag{1.4}$$

establishing a relationship between the three inequalities [DS].

In the study of hydrodynamic scaling limits of interacting particle systems, an understanding of the decay to equilibrium of the dynamics is an important ingredient. See for example [KL] for a review of the available methods. If the spectral gap is of the order N^{-2}, we can establish a hydrodynamic scaling limit for the process. However, when we consider the fluctuations of this result in *nonequilibrium*, the central limit theorem multiplicative re-scaling by $\sqrt{N}$ requires stronger tools. Chang and Yau [CY] developed a method to prove nonequilibrium density fluctuations for the Ginzburg-Landau model, which makes use of the logarithmic Sobolev inequality. Indeed, our main interest in the inhomogeneous inequality stemmed from the study of the nonequilibrium fluctuations of the hydrodynamic scaling limit for the colour version of the zero range process.

Spectral bounds for the zero range process have been studied extensively in the literature. The spectral gap of order N^{-2} was established by Landim, Sethuraman and Varadhan for the homogeneous symmetric nearest neighbour zero range process [LSV]. They make the usual assumptions on the jump rate function of the zero range process, $c(\cdot)$. Namely, they assume Lipschitz growth of the rate function

$$\sup_k |c(k+1) - c(k)| < \infty \tag{1.5}$$

as well as a weak monotonicity condition:

$$\inf_k \{c(k+k_0) - c(k)\} > 0, \tag{1.6}$$

for some integer k_0. Assumption (1.5) is necessary to ensure that the zero range process is well defined on the infinite lattice [A]. Condition (1.6) rules out the cases, such as the queueing system corresponding to $c(k) = \mathbb{I}(k \ge 1)$, where C_{SG} depends on the density of particles.

These are also the assumptions under which Dai Pra and Posta showed the logarithmic Sobolev inequality in [DPP1, DPP2]. There they show that $C_{\mathrm{LS}} = CN^2$, where C is independent of the particle density. Their approach is based on the martingale method of Lu and Yau [LY].

Recently, Caputo and Posta [CP] studied the case of the inhomogeneous zero range process on the complete graph. As before, allow the system to evolve on a cube of size N^d. The complete graph setting means that particles are allowed to jump to any other location of the cube with equal probability. In the nearest neighbour case, also known as local dynamics, particles make jumps to one of their nearest neighbours. Inhomogeneity means the the rate at which the first particle leaves site x depends on x, and hence we now consider a family of rate functions $c_x(\cdot)$.

For the complete graph dynamics it is shown in [CP] that under the condition

$$\inf_{x,\,k}\{c_x(k+1)-c_x(k)\}>0$$

on the rate functions, the system has a spectral gap of constant order. This is known to imply a spectral gap of order N^{-2} for the nearest neighbour model [Q]. Under the additional assumption

$$\sup_{x,\,k}\{c_x(k+1)-c_x(k)\}<\infty,$$

Caputo and Posta also prove the entropy dissipation inequality (1.3) for the complete model.

In this article we consider the symmetric nearest neighbour inhomogeneous zero range process. Under these dynamics, the particles move around a cubic subset of $\mathbb{Z}^d$ of size N^d. Particles wait exponential time to make a jump, and then jump to one of their closest neighbours with equal probability. We show that in this case the logarithmic Sobolev constant $C_{\rm LS}$ behaves like CN^2. It seems natural to study the problem under the uniform versions of conditions (1.5) and (1.6)

$$\begin{aligned}(LG)\quad & \sup\nolimits_{k,\,x}|c_x(k+1)-c_x(k)|\le a_1<\infty\\ (M)\quad & \inf\nolimits_{k,\,x}\{c_x(k+k_0)-c_x(k)\}\ge a_2>0,\end{aligned}$$

for some fixed constants a_1, a_2 and integer k_0. However, for technical reasons, at this time we also need to make the additional condition we now describe.

In the homogeneous case the grand canonical measures are product measures with marginals indexed by the *constant* density ρ. When we move to the inhomogeneous case this is no longer true; the marginals are not spatially homogeneous. However, we may now index the measure by the *overall* density, which we define as the average of the local densities ρ_x. Let $\mu_{\Lambda,\rho}$ denote the grand canonical measure for the process on the box Λ with overall density ρ, and the zero range particle configurations by η. We assume that

$$0<\inf_r\sqrt{r}\,\mu_{\Lambda,\frac{r}{|\Lambda|}}\left(\sum_{x\in\Lambda}\eta(x)=r\right)\le\sup_r\sqrt{r}\,\mu_{\Lambda,\frac{r}{|\Lambda|}}\left(\sum_{x\in\Lambda}\eta(x)=r\right)<\infty \tag{1.7}$$

for any size $|\Lambda|\ge 2$. This condition is required only in the proof of Lemma 5.6. One simple case when this condition is satisfied occurs when we assume that there exists a positive constant θ and universal K_0 such that

$$c_x(k)=\theta k,\quad \forall\ x \text{ and } k\ge K_0. \tag{1.8}$$

See Remark 5.7 for more details.

The inhomogeneous zero range process arises naturally in the following setting. Consider first the homogeneous case and assign one of k colours to each of the particles. The configurations of particles of each colour, considered jointly, form a Markov process with a family of invariant measures. Next, we single out one of the colours,

and condition on the configuration of the remaining particles. The invariant measures are still product measures which can be seen as a special case of the invariant measures for the inhomogenous model. One can easily show that assumptions (1.5) and (1.6) on the rate function $c(\cdot)$ in the original homogeneous process imply conditions (LG) and (M) for the induced inhomogeneous rates. The additional condition can be attained if we assume, for example, that $c(k) = \theta k$ for all k sufficiently large, as this implies (1.8), which in turn implies (1.7).

The relationship of the inhomogeneous process with the colour homogeneous version was our main interest in writing down this result. Notice that the system evolution for one colour in the colour version of the process is *not* the same as the evolution of one colour conditioning on, or fixing, the remaining particles. However, because the logarithmic Sobolev inequality is a *static* property, the inhomogeneous setting provides useful spectral bounds regardless. Using this idea, we were able to establish nonequilibrium fluctuations for the colour zero range process [J].

The proof of (1.1) for the inhomogenous zero range process, with $C_{\mathrm{SG}} = CN^2$, is the main result of this paper. Our approach is a direct extension of the work of Dai Pra and Posta [DPP1, DPP2] for the homogeneous version.

2. Notation and Main Results

Throughout this paper we shall use the following notation to denote the mean and covariance on a probability triple $(\Omega, \mathcal{F}, \mu)$:

$$\mu[f] := \int f d\mu, \quad \mu[f;g] := \mu[(f-\mu[f])(g-\mu[g])].$$

For a sub-σ-field $\mathcal{G}$ of $\mathcal{F}$, the conditional mean and covariance is defined similarly by

$$\begin{aligned} \mu[f|\mathcal{G}] &= \int f(\cdot)\mu(\cdot|\mathcal{G}), \\ \mu[f;g|\mathcal{G}] &= \mu[(f-\mu[f|\mathcal{G}])[g-\mu[g|\mathcal{G}]]|\mathcal{G}]. \end{aligned}$$

The entropy $H(f|\mu)$ is defined as

$$H(f|\mu) = \mu[f\log f] - \mu[f]\log\mu[f]$$

for a nonnegative f, and we may sometimes use the notation $H_\mu(f)$. Notice that for a density f the entropy simply becomes $\mu[f\log f]$. The conditional entropy is defined as

$$H[f|\mathcal{G}] = \mu[f\log f|\mathcal{G}] - \mu[f|\mathcal{G}]\log\mu[f|\mathcal{G}].$$

Given a function h and a set Λ, we will write

$$AV_{z\in\Lambda}h(\eta(z))$$

to denote the sample average of $h(\eta(z))$; that is, $\frac{1}{|\Lambda|}\sum_{z\in\Lambda} h(\eta(z))$.

Nearest Neighbour Inhomogeneous Zero Range. The inhomogeneous zero range process is a continuous time Markov process where particles perform random

walks with varying rates. The particles move around some subset Λ of $\mathbb{Z}^d$, and the rate at which particles make a jump depends on the total number of particles at the same site. Thus the particles form a system of continuous-time interacting random walks. The name "zero range" comes from the notion that each particle is interacting only with the particles at the same site, and hence the interaction has "no range".

We are interested in the evolution of the number of particles at each site. To this end, let $\eta(x)$, denote the number of particles at site x in Λ. The function η is an element of the space $\mathcal{X} = \mathbb{N}^\Lambda$. To indicate that we are referring to the function η restricted to some subset $\tilde{\Lambda} \subset \Lambda$ we will use the notation $\eta_{\tilde{\Lambda}}$. For each $x \in \Lambda$ fix a rate function $c_x : \mathbb{N} \mapsto \mathbb{R}$ such that $c_x(0) = 0$ and it is strictly positive otherwise. A particle at site x waits independently for an exponential amount of time with rate $c_x(\eta(x))/\eta(x)$ and then jumps from its current position to one of its nearest neighbours y. To maintain symmetry the particle chooses either neighbour with equal probability. Note that this implies that the first particle to jump from site x does so at rate $c_x(\eta(x))$.

If a particle moves from site x to site y the configuration η changes to $\eta^{x,y}$ where

$$(\eta^{x,y})(z) \;=\; \begin{cases} \eta(z) - 1 & \text{if } z = x, \\ \eta(z) + 1 & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

Again, this change occurs at rate $c_x(\eta(x))$. The time evolution of the configuration η forms a Markov process and we may write its generator L as

$$\begin{aligned} (Lf)(\eta) &= \frac{1}{2} \sum_{x \sim y \in \Lambda} c_x(\eta(x))[f(\eta^{x,y}) - f(\eta)] \\ &= \frac{1}{2} \sum_{x \sim y \in \Lambda} c_x(\eta(x)) \nabla_{x,y} f \end{aligned} \tag{2.1}$$

where $x \sim y$ denotes nearest neighbours of $\mathbb{Z}^d$ (or Λ).

Notice that the dynamics we have described preserve the total number of particles as the system evolves through time. For a configuration η, we shall denote the total number of particles as $R = R(\eta)$, and a realization of this random variable as r. Thus, for the case where Λ is finite and the total number of particles is r, the dynamics describe an irreducible Markov process on a finite state space $\mathcal{X}_r = \{\eta \in \mathcal{X} | R = r\}$. The stationary measure for this process is denoted by $\nu_{\Lambda,r}$, and is proportional to

$$\Pi_{x \in \Lambda} \frac{1}{c_x(\eta(x))!},$$

where we define the factorial $c_x(k)!$ to be $c_x(k) \times c_x(k-1) \times \cdots \times c_x(1)$, with the convention that $c_x(0)! = 1$.

The canonical ensembles $\nu_{\Lambda,r}$ satisfy the detailed balance condition, and the system is hence reversible. That is, whenever $\eta(x) > 0$, we have

$$c_x(\eta(x))\nu_{\Lambda,r}(\eta) = c_y(\eta(y)+1)\nu_{\Lambda,r}(\eta^{x,y}). \tag{2.2}$$

This allows us to write the Dirichlet form $D_{\Lambda,r}(f) = \nu_{\Lambda,r}[f(-L)f]$ in the more convenient form

$$D_{\Lambda,r}(f) = \frac{1}{2}\sum_{x\sim y\in\Lambda} \nu_{\Lambda,r}[c_x(\eta_x)\{\nabla_{x,y}f\}^2].$$

We next consider the grand canonical measures. To this end, fix φ in $(0,\infty)$, and define $\mu_{\Lambda,\varphi}$ as the product measure with marginals

$$\mu_{\{x\},\varphi}(\eta_x = k) = \frac{\varphi^k}{c_x(k)!\, Z_x(\varphi)},$$

where $Z_x(\varphi) = \sum_{k\geq 0} \frac{\varphi^k}{c_x(k)!}$ is the normalizing factor. $Z_x(\varphi)$ is also called the partition function. The grand canonical measures continue to satisfy the detailed balance condition. The canonical ensembles are equal to the grand canonical measure conditioned on the total number of particles, $\nu_{\Lambda,r}(\eta) = \mu_{\Lambda,\varphi}(\eta|R=r)$.

Let ρ_x denote the density of particles at site x, $\rho_x = \mu_{\Lambda,\varphi}(\eta_x)$. We use the notation ρ_Λ to denote the average of the function ρ_x,

$$\rho_\Lambda = \frac{1}{|\Lambda|}\sum \rho_x.$$

For a fixed Λ, we shall often simplify this notation to ρ. Note that the identity $\varphi = \mu_{\Lambda,\varphi}[c_x(\eta_x)]$ continues to hold for the inhomogeneous system. Also, $\rho = \rho(\varphi)$, considered as a function of φ, is strictly increasing. We will use the notation $\sigma_x^2 = \sigma_x^2(\varphi)$ to denote the variance at η_x, $\mu_{\Lambda,\varphi}[\eta_x;\eta_x]$. We also define

(i) the average density

$$\sigma_\Lambda^2(\varphi) = \frac{1}{|\Lambda|}\sum \sigma_x^2(\varphi),$$

(ii) the k^{th} moment at site x, $m_k^x = E_{\mu_{\{x\},\varphi}}[(\eta_x - \rho_x)^k]$, and

(iii) the Fourier transform of the marginal

$$\hat{\mu}_\varphi^x(t) = E_{\mu_{\{x\},\varphi}}[e^{it(\eta_x-\rho_x)/\sigma_x}].$$

For the homogeneous model , it is standard practice to index the product measures $\mu_{\Lambda,\varphi}$ by ρ instead of φ. This is natural as there exists a one-to-one map between the two quantities, and ρ may be interpreted as the density, a quantity easily seen to be preserved by the system dynamics. However, we will not do this here. Nonetheless, an invertible relationship continues to hold in our setting. For any fixed set Λ, there is a one-to-one relationship between ρ and $\varphi = \varphi(\rho)$. We may then, using the canonical measure, fully recover the function ρ_x.

As stated in the introduction, we shall assume that the rate functions $c_x(\cdot)$ satisfy Lipschitz growth (LG) and weak monoticity (M), *uniformly* in x. The two conditions imply that there exist universal constants c_1 and c_2 so that

$$0 < c_1 \leq \frac{c_x(k)}{k} \leq c_2 < \infty \tag{2.3}$$

for all k and x. Additionally, we assume

$$(E) \qquad 0 < \inf_r \sqrt{r}\, \mu_{\Lambda,\varphi(\frac{r}{|\Lambda|})} \left(\sum_{x \in \Lambda} \eta(x) = r \right) \leq \sup_r \sqrt{r}\, \mu_{\Lambda,\varphi(\frac{r}{|\Lambda|})} \left(\sum_{x \in \Lambda} \eta(x) = r \right) < \infty$$

for any size $|\Lambda| \geq 2$. This is simply a restatement of (1.7) using the notation developed above. Again, we direct the reader to Remark (5.7) for further details.

We are finally in the position to state the main result.

Theorem 2.1. *Assume that conditions (LG),(M) and (E) are satisfied by the inhomogeneous zero range process. Then the system defined on $\Lambda \subset \mathbb{Z}^d$, a cube of size N^d, satisfies a logarithmic Sobolev inequality with logarithmic constant of the order of N^2. That is, there exists a constant $C > 0$ such that for any choice of r, $|\Lambda| \geq 2$, and non-negative function f,*

$$H(f|\nu_{\Lambda,r}) \leq C N^2 D_{\Lambda,r}(\sqrt{f}).$$

The constant C may depend on the dimension d of the cube, but it is constant in N as well as r, the total number of particles.

Remark 2.2. *The constant C in the above inequality depends also on the parameters of the model given by the assumptions (LG), (M) and (E). However, we choose not to keep track of the exact form of the dependence.*

As we mentioned in the introduction, the logarithmic Sobolev inequality implies the spectral gap. However, the proof of Theorem 2.1 makes use of the spectral gap for the zero range process, and so it was necessary to prove the following beforehand.

Theorem 2.3 (Spectral Gap)**.** *Assume that conditions (LG) and (M) hold uniformly for the inhomogeneous zero range process. Then for the system defined on $\Lambda \subset \mathbb{Z}^d$, where $|\Lambda| = N^d$, there exists a finite constant $C^* > 0$, such that*

$$\nu_{\Lambda,r}[f; f] \leq C^* N^2 D_{\Lambda,r}(f)$$

for all $f \geq 0$. The constant C^* may depend on the dimension d, as well as the constants a_1, a_2 and k_0 of the assumptions.

Notice that for this result we do not need the additional assumption (E).

Connection to Zero Range with Colours. To simplify notation we define the k-colour model for the case when $k = 2$. The extension to general k is obvious.

First, consider the colour-less (or *colour-blind*) homogeneous zero range process. To define this, simply take the previously described inhomogeneous system and add the requirement that the jump rates satisfy $c_x(\cdot) = c(\cdot)$ for all x. The invariant measures now become the product measures $\mu_{\Lambda,\varphi}$ with spatially homogeneous marginals

$$\mu_\varphi(\eta(x) = k) = \frac{\varphi^k}{c(k)!} Z^{-1}(\varphi). \tag{2.4}$$

$Z(\varphi)$ is again the normalizing factor.

Next, imagine that this process is made up of two different colours of particles. The particles are mechanically identical to the regular zero range particles, but we now also keep track of their colour as the system evolves. Let $\eta_i(x)$ denote the number of particles of colour i at site x in Λ. Notice that the time evolution of $\boldsymbol{\eta} = \{\eta_1, \eta_2\}$ is a Markov process with state space $\mathcal{X}^2 = \mathbb{N}^\Lambda \times \mathbb{N}^\Lambda$. Define the colour rate functions

$$\begin{aligned} c^1(k_1, k_2) &= k_1 \frac{c(k_1 + k_2)}{k_1 + k_2}, \\ c^2(k_1, k_2) &= c(k_1 + k_2) - c^1(k_1, k_2). \end{aligned}$$

The generator for the two-colour process is then

$$(L^{\text{colour}} f)(\boldsymbol{\eta}) = \sum_{i=1}^{2} \sum_{x \sim y} c^i(\boldsymbol{\eta}(x))[f(\boldsymbol{\eta}_i^{x,y})) - f(\boldsymbol{\eta})], \tag{2.5}$$

where $\boldsymbol{\eta}_i^{x,y}$ denotes the configuration obtained from $\boldsymbol{\eta}$ by moving one particle of colour i from site x to site y. Note that if the function f is "blind" to the particle colour, i.e. $f(\boldsymbol{\eta}) = f(\eta_1 + \eta_2)$, then $L^{\text{colour}} f$ is equivalent to the generator for the homogeneous zero range process.

Fix $0 < \varphi_1, \varphi_2 < \infty$. The grand canonical measures for the two colour process are the product measures with marginals

$$\mu_{\varphi_1,\varphi_2}(\eta_1(x) = k, \eta_2(x) = m) = \binom{k+m}{k} \frac{\varphi_1^k \ \varphi_2^m}{c(k+m)!} \quad Z^{-1}(\varphi_1 + \varphi_2),$$

where $Z(\varphi)$ is the same partition function as in (2.4). Further calculations show that

$$\begin{aligned} \mu_{\varphi_1,\varphi_2}(\eta_1(x) = k | \eta_2(x) = 0) &= \frac{\varphi_1^k}{c(k)!} Z^{-1}(\varphi_1) \\ \mu_{\varphi_1,\varphi_2}(\eta_1(x) + \eta_2(x) = n) &= \frac{\varphi^n}{c(n)!} Z^{-1}(\varphi_1 + \varphi_2) \\ \mu_{\varphi_1,\varphi_2}(\eta_1(x) = k, \eta_2(x) = n-k | \eta(x) = n) &= \binom{n}{k} \left(\frac{\varphi_1}{\varphi_1 + \varphi_2} \right)^k \left(\frac{\varphi_2}{\varphi_1 + \varphi_2} \right)^{n-k}. \end{aligned}$$

Notably, we obtain that

$$\mu_{\varphi_1,\varphi_2}(\eta_1(x) = k | \eta_2(x) = m) = \frac{\varphi_1^k}{c_m(k)!} Z_m^{-1}(\varphi_1),$$

with

$$c_m(k) = \frac{kc(k+m)}{k+m}$$

and we denote the associated partition function as $Z_m(\cdot)$. Equivalently, we can say that the grand canonical measures for the first colour conditioned on the configuration of the second colour are product measures with marginals

$$\tilde{\mu}(\eta_1(x) = k) = \frac{\tilde{\varphi}^k}{\tilde{c}_x(k)!}\tilde{Z}_x^{-1}(\tilde{\varphi}),$$

where $\tilde{\varphi} = \varphi_1$,

$$\tilde{c}_x(k) = k\,\frac{c(k+\eta_2(x))}{k+\eta_2(x)},$$

and $\tilde{Z}_x(\tilde{\varphi})$ is the normalizing constant. We hence obtain the invariant measures for an inhomogeneous process. It is not difficult to show that assuming conditions (1.5) and (1.6) imply uniform (LG) and (M) conditions for the rates $\tilde{c}_x(\cdot)$. Assuming that, for example, $c(k) = \theta k$ for all sufficiently large k, gives also the additional condition (E).

Notice that this relationship holds regardless of the number of colours originally considered.

3. Outline of Proof

The proof of Theorem 2.1 is divided into two main sections. First, we prove that the logarithmic Sobolev constant is independent of the number of particles r. Second, we obtain sharper bounds which give the desired N^2 scaling for large enough N. The combination of these two results implies Theorem 2.1. In each case we use induction in the size of Λ, where each induction step doubles the size of the cube. For ease of presentation, we write out the proof for the case when $d = 1$. Similar arguments as those presented in the proof of the spectral gap (Section 8) extend the argument to the general case.

Suppose then that $|\Lambda| = 2N$, so that we may write $\Lambda = \Lambda_1 \cup \Lambda_2$ where $|\Lambda_1| = |\Lambda_2| = N$, and the two subsets are disjoint. We shall denote the number of particles on η_{Λ_i} as R_i, with $R = R_1 + R_2$. We thus have

$$\begin{aligned} H(f|\nu_{\Lambda,r}) &= \nu_{\Lambda,r}[H(f|\nu_{\Lambda,r}(\cdot|R_1 = r_1))] \\ &\qquad + H(\nu_{\Lambda,r}[f|R_1 = r_1]|\nu_{\Lambda,r}) \\ &\leq \nu_{\Lambda,r}[H(f|\nu_{\Lambda_1,r_1})] + \nu_{\Lambda,r}[H(f|\nu_{\Lambda_2,r-r_1})] \\ &\qquad + H(\nu_{\Lambda,r}[f|R_1 = r_1]|\nu_{\Lambda,r}) \end{aligned} \tag{3.1}$$

because $\nu_{\Lambda,r}(\cdot|R_1 = r_1) = \nu_{\Lambda_1,r_1} \otimes \nu_{\Lambda_2,r-r_1}$. Let $\kappa(N,r)$ be the smallest constant such that

$$H(f|\nu_{\Lambda',r'}) \leq \kappa(N,r) D_{\Lambda',r'}(\sqrt{f}) \tag{3.2}$$

for all volumes $|\Lambda'| \leq N$ and $r' \leq r$ particles. The induction hypothesis provides us with the bound on the first half of (3.1) (recall that here $|\Lambda| = 2N$)

$$\begin{aligned} H(f|\nu_{\Lambda,r}) &\leq \kappa(N,r)\nu_{\Lambda,r}\left[D_{\Lambda_1,r_1}(\sqrt{f}) + D_{\Lambda_2,r-r_1}(\sqrt{f})\right] \\ &\qquad + H(\nu_{\Lambda,r}[f|R_1 = r_1]|\nu_{\Lambda,r}) \\ &\leq \kappa(N,r)D_{\Lambda,r}(\sqrt{f}) + H(\nu_{\Lambda,R}[f|R_1 = r_1]|\nu_{\Lambda,r}). \end{aligned} \tag{3.3}$$

It thus remains to estimate the last term to obtain the appropriate bounds.

In Section 6, we prove the following initial bound

$$H(\nu_{\Lambda,R}[f|R_1 = r_1]|\nu_{\Lambda,r}) \leq C(N)(1 + \kappa(N,r))D_{\Lambda,r}(\sqrt{f}) + C(N)\nu_{\Lambda,r}[f], \tag{3.4}$$

where $C(N)$ is a constant depending only on N. Together with (3.3) this implies that

$$H(f|\nu_{\Lambda,r}) \leq C(N)\left(\nu_{\Lambda,r}[f] + D_{\Lambda,r}(\sqrt{f}) + \kappa(N,r)D_{\Lambda,r}(\sqrt{f})\right). \tag{3.5}$$

Define the function $\tilde{f} = (\sqrt{f} - \nu_{\Lambda,r}[\sqrt{f}])^2$. In Lemma 4.1 we give a proof of the inequality

$$H(f|\nu_{\Lambda,r}) \leq H(\tilde{f}|\nu_{\Lambda,r}) + 2\nu_{\Lambda,r}[\sqrt{f};\sqrt{f}], \tag{3.6}$$

due to Rothaus [R]. This, together with (3.5) evaluated at $\tilde{f}$, gives the new bound

$$H(f|\nu_{\Lambda,r}) \leq C(N)\left(\nu_{\Lambda,r}[\sqrt{f};\sqrt{f}] + D_{\Lambda,r}(\sqrt{f}) + \kappa(N,r)D_{\Lambda,r}(\sqrt{f})\right).$$

By applying the spectral gap result of Theorem 2.3 we may bound this again by

$$H(f|\nu_{\Lambda,r}) \leq C(N)\left(D_{\Lambda,r}(\sqrt{f}) + \kappa(N,r)D_{\Lambda,r}(\sqrt{f})\right).$$

for some different constant $C(N)$. From this it follows that

$$\kappa(2N,r) \leq C(N)(1 + \kappa(N,r)).$$

In Section 6 we prove the initial induction step

$$\sup_r \kappa(2,r) < \infty. \tag{3.7}$$

These two bounds imply that the logarithmic Sobolev constant is independent of r, the number of particles.

The second part in the proof consists of tightening (3.4) above to

$$H(\nu_{\Lambda,R}[f|R_1 = r_1]|\nu_{\Lambda,r}) \leq CN^2 D_{\Lambda,r}(\sqrt{f}) + C\nu_{\Lambda,r}[f] + \kappa(N,r)D_{\Lambda,r}(\sqrt{f}) \tag{3.8}$$

for some $C > 0$ and all large enough N. After eliminating the term $\nu_{\Lambda,r}[f]$ as before, this allows us to conclude that for large enough N we have the relationship

$$\kappa(2N,r) \leq \kappa(N,r) + CN^2,$$

for some new constant C. From this, by induction, we conclude Theorem 2.1.

The tighter bounds are discussed in section 7. The better estimates are obtained by improved bounds on the covariances appearing in Proposition 6.5 of Section 6.

These are possible because of the local limit theorems established in Section 4. The local limit theorems and resulting moment bounds are the main tools established in this section.

Section 5 looks at the spectral gap and logarithmic Sobolev inequality for several birth and death processes. These are key in establishing the bounds on the second half of (3.3).

In section 8, we prove Theorem 2.3 for any dimension d. As mentioned previously, a similar approach generalizes the arguments of Sections 6 and 7 to higher dimensions.

Remark on notation. In what follows, for reasons of presentation, we use the notation η_x in lieu of $\eta(x)$. As we no longer deal with the colour version of zero range, there is no inconsistency to resolve.

4. Preliminary Results

In this section we establish certain tools used throughout this paper. We give a proof of (3.6) and discuss stochastic monotonicity on the canonical ensembles which will be used in Section 5. Most notably, we establish uniform local limit theorems for very small, small and large values of the parameter φ. These results are also used to give several bounds on the zero range moments.

Lemma 4.1. *Let $\mathcal{X}$ denote a Polish space. For any nonnegative function $\phi : \mathcal{X} \mapsto \mathbb{R}$ and probability measure μ on $\mathcal{X}$ the following inequality holds:*

$$H(\phi|\mu) \le H(\tilde{\phi}|\mu) + 2\mu[\sqrt{\phi}; \sqrt{\phi}\,],$$

where $\tilde{\phi} = (\sqrt{\phi} - \mu[\sqrt{\phi}\,])^2$.

Proof. It is enough to consider continuous bounded functions f. The inequality follows if we can prove for any such f and constant k:

$$H(f^2|\mu) + 2\mu[f; f] \ge H((f+k)^2|\mu). \tag{4.1}$$

The proof of this for the case of μ Lebesgue measure appears in [R]. To begin we define the following quantities

$$\tilde{H}_k(t) = H((tf+k)^2|\mu)$$

and

$$\tilde{C}(t) = \mu[tf; tf].$$

We calculate the first two derivatives of the $\tilde{H}_k(t)$ to be

$$\partial_t \tilde{H}_k(t) = \int \{2(tf+k)f \cdot \log(tf+k)^2\} d\mu - \int \{2(tf+k)f\} d\mu \cdot \log \int (tf+k)^2 d\mu$$

and

$$\begin{aligned}\partial_t^2 \tilde{H}_k(t) &= \int \{2f^2 \log(tf+k)^2\} d\mu - \int 2f^2 d\mu \cdot \log \int (tf+k)^2 d\mu \\ &\quad + \int (2f)^2 d\mu - \frac{(\int \{2(tf+k)f\} d\mu)^2}{\int (tf+k)^2 d\mu}.\end{aligned}$$

Notice that in the above there is a potential problem whenever $(tk+f)=0$; in this case we may make the convention that $\log(tk+f)=0$ and the above still integrates to the correct thing.

Notice also that we have

$$\tilde{H}_k(t)\Big|_{t=0} = 0 \qquad \text{and} \qquad \partial_t \tilde{H}_k(t)\Big|_{t=0} = 0, \tag{4.2}$$

and that the same holds for the function $\tilde{C}(t)$. Fix the function f and the constant k and let $\tilde{H}(t) = \tilde{H}_k(t)|_{k=0}$. Inequality (4.1) may be re-written as

$$\tilde{H}(t) + 2\tilde{C}(t) \geq \tilde{H}_k(t),$$

for all $t \geq 0$. The inequality holds at $t=0$ and also after taking one derivative in t and evaluating at $t=0$ by (4.2), as in both cases both sides are simply zero. Hence, by integrating twice, we will obtain (4.1) if we can show that

$$\partial_t^2 \tilde{H}(t) + 2\partial_t^2 \tilde{C}(t) \geq \partial_t^2 \tilde{H}_k(t),$$

holds for all t. By our previous calculations this is the same as showing the following holds:

$$\begin{aligned}&\int \{2f^2 \log(tf)^2\} d\mu - \int 2f^2 d\mu \cdot \log \int (tf)^2 d\mu \\ &\qquad \geq \int \{2f^2 \log(tf+k)^2\} d\mu - \int 2f^2 d\mu \cdot \log \int (tf+k)^2 d\mu \\ &\qquad\quad - \frac{(\int \{2(tf+k)f\} d\mu)^2}{\int (tf+k)^2 d\mu}.\end{aligned}$$

This last inequality holds because by the variational formula of the entropy we may deduce that for any f and g we have that

$$\int f^2 \log f^2 d\mu - \int f^2 d\mu \cdot \log \int f^2 d\mu \geq \int f^2 \log g^2 d\mu - \int f^2 d\mu \cdot \log \int g^2 d\mu. \tag{4.3}$$

Indeed, it is enough to consider f such that $\int f^2 d\mu = 1$. Recall that the variational formula of entropy is

$$H(\phi|\mu) = \sup_{h \in C_b(\mathcal{X})} \left\{ \int h\, \phi\, d\mu - \log \int e^h d\mu \right\}.$$

To obtain (4.3) we need only choose $h = \log g^2$. $\square$

The following result is proved for the homogeneous model in [LSV]. It remains valid for the inhomogeneous model.

Lemma 4.2. *There exists a constant $B = B(a_1, a_2, k_0)$ such that*

$$\nu_{\Lambda,r} \leq \nu_{\Lambda,r+M},$$

for all $M \geq B|\Lambda|$.

This is an equivalent statement to the following theorem (see, for example, [L]).

Lemma 4.3. *There exists a constant $B = B(a_1, a_2, k_0)$ such that if $M \geq B|\Lambda|$ there exists a measure Q on $\mathcal{X} \times \mathcal{X}$ which is concentrated on the configurations $\{\eta, \xi\}$ with $\eta \leq \xi$, that is,*

$$Q(\eta \leq \xi) = 1,$$

and the marginals of Q are $\nu_{\Lambda,r}$ and $\nu_{\Lambda,r+M}$.

Proof. To clarify the proof we assume that the $k_0 = 2$ in assumption (M), that is, $c_x(k) - c_x(j) \geq a_2$ for $k - j \geq 2$. Consider the following version of the complete zero range process with generator

$$\mathcal{L} = \sum_{x,y \in \Lambda} c_x(\eta_x)[\nabla_{x,y} f].$$

The measures $\nu_{\Lambda,r}$ are ergodic and reversible for the complete process as well.

Consider two configurations η and ξ where $R(\eta) = r$ and $R(\xi) = r + M$ and $\eta \leq \xi$. Our goal is to define a coupled process $\{\eta(t), \xi(t)\}$ with initial configuration $\{\eta, \xi\}$, which preserves the order $\eta(t) \leq \xi(t)$ at all times t and where the marginals of $\eta(t)$ and $\xi(t)$ evolve according to the dynamics defined by the generator $\mathcal{L}$. To this end we define the subsets of Λ

$$b_0 = \{x; \eta_x - \xi_x = 0\} \text{ and } b_1 = \{x; \xi_x - \eta_x = 1\}.$$

We may now begin to consider the coupling. We need to describe the jumps of the η and ξ particles so that for any two configurations η and ξ with $\eta \leq \xi$ the order is preserved after any possible jump.

First we make the particles on the sites b_0 make a jump together. Because $c_x(k) - c_x(j) \geq a_2$ for $k - j \geq 2$ for all sites of $\Lambda \setminus b_0 \cup b_1$ we may couple all of the η particles with a ξ particle, and we are left with $M - |b_1|$ "free" ξ particles. The only problem in the definition of a coupling occurs if an η particle jumps from a b_1 site to a b_0 site. This event occurs at rate of at most $a_1|b_0||b_1|$. We may compensate for this behaviour using our "free" ξ particles. The rate at which these particles make a jump to b_0 is

$$b_0 \sum_{x \in \Lambda \setminus b_0 \cup b_1} c_x(\xi_x) - c_x(\eta_x)$$

which is bounded below by

$$\begin{aligned} b_0 \sum \left\{ a_2 \lfloor \frac{\xi_x - \eta_x}{k_0} \rfloor - a_1 k_0 \right\} &\geq b_0 \sum \left\{ a_2 \left(\frac{\xi_x - \eta_x}{k_0} - 1 \right) - a_1 k_0 \right\} \\ &\geq \frac{a_2 b_0}{k_0}(M - |b_0|) - (a_2|b_0| + a_1 k_0)|\Lambda \setminus b_0 \cup b_1| \\ &\geq \frac{a_2 b_0}{k_0} M - b_0 |\Lambda| \left\{ a_2 + \frac{a_2}{k_0} + a_1 k_0 \right\}. \end{aligned}$$

This will be greater than $a_1|b_0||b_1|$ as long as $M \geq B|\Lambda|$ for $B = \frac{a_1}{a_2} k_0(k_0+1) + k_0 + 1$. Finally, because the rate at which the "free" ξ particles jump to b_0 sites is greater than the rate at which the uncoupled η particles do so, we may couple these jumps while preserving the correct marginal dynamics. All other particles are allowed to jump freely. This is exactly what we need so that our joint process preserves the order $\eta(t) \leq \xi(t)$ at all times t. □

4.0.1. *Local Limit Theorems.* We begin with some moment bounds.

Proposition 4.4. (i) *For all x in Λ*

$$0 < c_1 \leq \frac{\varphi}{\rho_x} \leq c_2 < \infty.$$

(ii) *There exist constants $0 < \tilde{c}_1 \leq \tilde{c}_2 < \infty$ such that for all x in Λ*

$$\tilde{c}_1 \leq \frac{\sigma_x^2(\varphi)}{\varphi} \leq \tilde{c}_2.$$

The constants $\tilde{c}_1$ and $\tilde{c}_2$ depend only on the values a_1, a_2 and k_0.

The first inequality is a simple consequence of (2.3). The proof of the second inequality appears in [LSV]. As the bounds are bounds on the single site marginals, they also apply in this setting. Because the bounds depend only on the constants a_1, a_2 and k_0 they apply uniformly to all x. Notice that we do not require the additional assumption (E). The same is true of the following:

Proposition 4.5. *For each $\varphi_0 > 0$ and $\bar{k}$ in $\mathbb{N}$*

(i) *There exists a finite constant K_0 such that*

$$\sup_{\varphi \geq \varphi_0} \frac{m_{2k}^x(\varphi)}{\sigma^{2k}(\varphi)} \leq K_0, \quad for\ 1 \leq k \leq \bar{k}.$$

(ii) *For every $\delta > 0$, there exists $C(\delta) < 1$ such that*

$$\sup_{\varphi \geq \varphi_0} \sup_{\delta \leq |t| \leq \pi\sigma(\varphi)} |\hat{\mu}_\varphi^x(t)| \leq C(\delta).$$

(iii) *There exists a $\kappa > 0$ so that*

$$\sup_{\varphi \geq \varphi_0} \int_{|t| \leq \pi\sigma(\varphi)} |\hat{\mu}_\varphi^x(t)|^\kappa \leq C < \infty.$$

From the above and a simple calculation we also obtain

Corollary 4.6. *There exists a finite, positive constant $\tilde{c}$ such that for all Λ*

$$\tilde{c}^{-1} \leq \varphi'(\rho) = \frac{\varphi}{\sigma^2(\rho)} \leq \tilde{c}.$$

We now turn to the local limit theorems. Recall the definition of the Hermite polynomial of degree m, for $m \geq 0$:

$$H_m(x) = (-1)^m \exp\left(\frac{x^2}{2}\right) \frac{d^m}{dx^m} \exp\left(-\frac{x^2}{2}\right).$$

Let $g_0(x)$ denote the density of a standard normal random variable, and, for $j \geq 1$, define

$$g_j(x) = g_0(x) \sum H_{j+2a}(x) \prod_{m=1}^{j} \frac{1}{k_m!} \left(\frac{\kappa_{m+2}}{(m+2)!\ \sigma^{m+2}} \right)^{k_m} \tag{4.4}$$

where the sum is taken over all nonnegative integer solutions $\{k_l\}_{l=1}^j$ and a of $k_1 + 2k_2 + \ldots + jk_j = j$ and $k_1 + k_2 + \ldots + k_j = a$, and κ_m denotes the k^{th} cumulant of the distribution.

In what follows, we assume that $|\Lambda| = N$.

Theorem 4.7. (i) *For all $\varphi_0 > 0$ and $J \in \mathbb{N}$, there exist finite constants $E_0 = E_0(\varphi_0, J)$ and $A = A(\varphi_0, J)$ such that*

$$\left| \sqrt{N\sigma^2} \mu_{\Lambda,\varphi} \left[\sum_{x \in \Lambda} \eta(x) = N\rho + \sigma\sqrt{N} z \right] - \sum_{j=0}^{J-2} \frac{1}{N^{j/2}} g_j(z) \right| \leq \frac{E_0}{(\sigma^2 N)^{(J-1)/2}}$$

uniformly over z and over all parameters $A/N \leq \varphi \leq \varphi_o$. In the above, $\sigma = \sigma_\Lambda(\varphi)$.

(ii) *For any $\varphi_0 > 0$ and any $k_1 \in \mathbb{N}$, there exists a constant $E_1 = E_1(k_1, \varphi_0)$ and $n_1 = n_1(k_1, \varphi_0)$ such that*

$$\left| \sqrt{N\sigma^2} \mu_{\Lambda,\varphi} \left[\sum_{x \in \Lambda} \eta(x) = N\rho + \sigma\sqrt{N} z \right] - \sum_{j=0}^{k_1-2} \frac{1}{N^{j/2}} g_j(z) \right| \leq \frac{E_1}{N^{(k_1-1)/2}}$$

for all $N > n_1$ uniformly over z and $\varphi > \varphi_0$. Again, in the above, $\sigma = \sigma_\Lambda(\varphi)$.

Sketch of Proof. The proof of this result is discussed in [LSV] for the homogeneous case. It relies on repeating the usual local limit theorem argument (see for example [GK] or [P]), while checking that the bounds are valid uniformly in the parameter φ

in the two settings. These bounds rely on moment estimates, which, due to conditions (LG) and (M) are identical in both the homogeneous and nonhomogeneous models.

Fix J in $\mathbb{N}$. Following the usual local limit theorem approach as in [P] we write

$$\sqrt{N\sigma^2}\mu_{\Lambda,\varphi}\left[\sum_{x\in\Lambda}\eta_x = N\rho+\sigma\sqrt{N}z\right] - \sum_{j=0}^{J-2}\frac{1}{N^{j/2}}g_j(z)$$

as

$$\int_{-\pi\sqrt{n}\sigma}^{\pi\sqrt{n}\sigma} e^{-itz}f_N(t)dt - \int e^{-itz}u_{k,N}(t)dt,$$

where $u_{k,N}(t) = \sum_{j=0}^{J-2}\frac{1}{N^{j/2}}\int e^{itx}g_j(x)dx$ and $f_N(t)$ is the characteristic function of $(\sqrt{N}\sigma)^{-1}\sum_{x\in\Lambda}(\eta_x-\rho_x)$. We may bound this quantity by the sum of 4 integrals

$$\begin{aligned} I_1 &= \int_{|t|<N^{1/6}} |f_N(t)-u_{k,N}(t)|dt \\ I_2 &= \int_{N^{1/6}\le|t|<T_N} |f_N(t)|dt \\ I_3 &= \int_{T_N\le|t|<\pi\sqrt{N}\sigma} |f_N(t)|dt \\ I_4 &= \int_{|t|>N^{1/6}} |u_{k,N}(t)|dt, \end{aligned}$$

where $T_N = \frac{1}{4}N^{3/2}\sigma^{3/2}\sum_{x=1}^N m_x^3$. The details on how to bound I_1, I_2 and I_4 are given in [P]: Theorem 12 of chapter 7 and Lemmas 11 and 12 of chapter 6. We obtain, for $i=1,2,4$, the bound

$$I_i \le \frac{C}{(\sigma^2(\varphi)N)^{(J-1)/2}},$$

where C is a positive constant not depending on φ. Notice that under the second regime we have that $\sigma^2(\varphi) > C\varphi_0$, for all $\varphi>\varphi_0$. The main difficulty lies in obtaining the appropriate uniform bounds on I_3. Because of the moment bounds at the beginning of this section we have for some b

$$|I_3| \le \sqrt{N}\sigma\int_{b<|t|<\pi}\Pi_{x\in\Lambda}|\mu_\varphi^x(t)|dt.$$

The result follows if we can show that $|\mu_\varphi^x(t)|$ stays strictly below 1 uniformly in φ and x. In part (1) of the theorem this follows from the following bound

$$|\mu_\varphi^x(t)|^2 - 1 \le C\varphi_0\varphi(\cos t - 1).$$

In part (2) of the theorem the necessary bound follows from part (2) of Proposition 4.5. This gives us exponential decay on the term I_3. □

The above are both Edgeworth expansions for a lattice distribution, which are valid uniformly for the family of measures μ_φ. Note that we require that the average density is not "too small". This should not be surprising, as in the case where $\rho < A/N$ there is at most a finite number of particles, and so the probability of observing a "success" decreases as $N \to \infty$ and a Poisson limit theorem holds instead of a Gaussian limit theorem.

Lemma 4.8. *For every $A \in \mathbb{N}\backslash\{0\}$ there exists a constant A_0 such that*

$$\sup_{r\leq A} \left| \mu_{\Lambda,\varphi_\Lambda(\frac{r}{N})} \left(\sum_{x\in\Lambda} \eta_x = k \right) - \frac{r^k}{k!} e^{-r} \right| \leq \frac{A_0}{N}$$

for any $k \in \mathbb{N}$, where $N = |\Lambda|$.

Proof. This lemma is proved in [DPP1] for the homogeneous system. We extend it here to the inhomogeneous case by using the general version of the Poisson limit theorem as proved in [B], for example.

$$\begin{aligned} \mu_{\Lambda,\varphi}(R = k) &= \mu_{\Lambda,\varphi}(R = k | \max \eta_x \leq 1) \mu_{\Lambda,\varphi}(\max \eta_x \leq 1) \\ &\quad + \mu_{\Lambda,\varphi}(R = k | \max \eta_x > 1) \mu_{\Lambda,\varphi}(\max \eta_x > 1) \end{aligned}$$

Since we choose $\varphi = \varphi_\Lambda(r/N)$ and because r is bounded above we have that $\mu_{\Lambda,\varphi}(\max \eta_x > 1) = O(\frac{1}{N})$. We thus need to show that

$$\mu_{\Lambda,\varphi}(R = k | \max \eta_x \leq 1) = \frac{r^k}{k!} e^{-r} + O(N^{-1}).$$

Define

$$\tilde{r}_x = \frac{\mu_{\Lambda,\varphi}[\eta_x \mathbb{I}_{\eta_x \leq 1}]}{\mu_{\Lambda,\varphi}(\eta_x \leq 1)},$$

and notice that this is equal to

$$\begin{aligned} \tilde{r}_x &= \frac{\rho_x - \mu_{\Lambda,\varphi}[\eta_x \mathbb{I}_{\eta_x \leq 1}]}{\mu_{\Lambda,\varphi}(\eta_x \leq 1)} \\ &= \frac{\rho_x + O(N^{-2})}{1 + O(N^{-2})}. \end{aligned} \tag{4.5}$$

Next consider the interval $[0, 1]$. Define $I_0(p) = [0, 1 - p]$ and $I_1(p) = [1 - p, 1]$. Also define $J_0 = [0, e^{-p}]$ and $J_m = [e^{-p} \frac{p^m}{m!}, e^{-p} \frac{p^{m+1}}{(m+1)!}]$ for $m \geq 1$. Notice that because $1 - p \leq e^{-p}$ we have

$$J_1(p) \subset I_1(p). \tag{4.6}$$

Consider a sequence of independent, Uniform $[0, 1]$ random variables U_x. Define

$$V_x = \begin{cases} 0 & \text{if } \; U_x \in I_0(\tilde{r}_x) \\ 1 & \quad \text{otherwise.} \end{cases}$$

Also define

$$W_x = i \text{ if } \quad U_x \in J_i(\tilde r_x).$$

First notice that in the system we have defined we have

$$P\left(\sum_{x\in\Lambda} V_x = k\right) = \mu_{\Lambda,\varphi}(R = k|\max \eta_x \le 1),$$

whereas

$$P\left(\sum_{x\in\Lambda} W_x = k\right) = e^{-\tilde r}\frac{(\tilde r)^k}{k!},$$

where we let $\tilde r = \sum_{x\in\Lambda} \tilde r_x$. Because of (4.5) we have

$$P\left(\sum_{x\in\Lambda} W_x = k\right) = e^{-r}\frac{r^k}{k!} + O(N^{-1}).$$

Also, because of (4.6), we have

$$\begin{aligned} P(W_x \ne V_x) &= P(U_x \in I_1(\tilde r_x) \setminus J_1(\tilde r_x)) \\ &\le \tilde r_x - e^{-\tilde r_x}\tilde r_x \le \tilde r_x^2. \end{aligned}$$

Thus

$$P\left(\sum_{x\in\Lambda} W_x \ne \sum_{x\in\Lambda} V_x\right) \le \sum_{x\in\Lambda} \tilde r_x^2 = O(N^{-1}).$$

The result follows once we note that

$$\begin{aligned} P\left(\sum_{x\in\Lambda} V_x = k\right) &= P\left(\sum_{x\in\Lambda} W_x = k\right) + P\left(\left\{\sum_{x\in\Lambda} V_x = k\right\}\bigcap\left\{\sum_{x\in\Lambda} W_x \ne \sum_{x\in\Lambda} V_x\right\}\right) \\ &- P\left(\left\{\sum_{x\in\Lambda} W_x = k\right\}\bigcap\left\{\sum_{x\in\Lambda} W_x \ne \sum_{x\in\Lambda} V_x\right\}\right). \end{aligned}$$

□

The above lemmas imply the following result. It follows by direct computation of the conditional probability and applying the bounds on the grand canonical measures obtained above.

Corollary 4.9. *Fix $\delta \in (0,1)$; there exists an n_0 and $A > 0$ such that for any $\Lambda_0 \subset \Lambda$ with $|\Lambda| > n_0$ and $\frac{|\Lambda_0|}{|\Lambda|} \le \delta$*

$$\nu_{\Lambda,r}(\eta_{\Lambda_0}) \le A\mu_{\varphi_\Lambda(\frac{r}{|\Lambda|})}(\eta_{\Lambda_0})$$

for all $r > 0$.

Proof. Let R_0 denote the total number of particles in Λ_0 and R_1 are the total number of particles in $\Lambda \setminus \Lambda_0$, respectively. We set $R_0(\eta_{\Lambda_0}) = r_0$. Let $|\Lambda| = N$ and $\varphi = \varphi_\Lambda(r/N)$. We begin by computing

$$\nu_{\Lambda,r}(\eta_{\Lambda_0}) = \mu_{\Lambda,\varphi}(\eta_{\Lambda_0}) \frac{\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\mu_{\Lambda,\varphi}(R = r)}.$$

We thus need to bound

$$\frac{\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\mu_{\Lambda,\varphi}(R = r)}$$

uniformly. This follows from the local limit theorems just described.

Case 1: $r \leq A$**.**

By Proposition 4.8 we have

$$\begin{aligned} \frac{\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\mu_{\Lambda,\varphi}(R = r)} &\leq \frac{1}{\mu_{\Lambda,\varphi}(R = r)} \\ &\leq \frac{1}{\frac{r^r e^{-r}}{r!} - \frac{A_0}{N}} \\ &\leq \frac{2}{\inf_{0 < r \leq A} \frac{r^r e^{-r}}{r!}}, \end{aligned}$$

for some large N_1 and all $N \geq N_1$.

Case 2: $A \leq r \leq \rho_0 N$**.**

In this case we write

$$\frac{\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\mu_{\Lambda,\varphi}(R = r)} = \sqrt{\frac{N}{N - |\Lambda_0|}} \frac{\sqrt{\sigma^2(N - |\Lambda_0|)}\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\sqrt{\sigma^2 N}\mu_{\Lambda,\varphi}(R = r)}.$$

By the first part of Theorem 4.7, setting $J = 2$, we have

$$\sqrt{\sigma^2(N - |\Lambda_0|)}\mu_{\Lambda,\varphi}(R_1 = r - r_0) \leq \frac{1}{\sqrt{2\pi}} + \frac{E_0}{\sqrt{\sigma^2(N - |\Lambda_0|)}} \leq \frac{1}{\sqrt{2\pi}} + \frac{E_0}{\sqrt{A(1-\delta)N}}$$

and

$$\sqrt{\sigma^2 N}\mu_{\Lambda,\varphi}(R = r) \geq \frac{1}{\sqrt{2\pi}} - \frac{E_0}{\sqrt{\sigma^2 N}} \geq \frac{1}{\sqrt{2\pi}} - \frac{E_0}{\sqrt{AN}}.$$

This implies that

$$\begin{aligned} \frac{\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\mu_{\Lambda,\varphi}(R = r)} &\leq \frac{1}{\sqrt{1-\delta}} \frac{\frac{1}{\sqrt{2\pi}} + \frac{E_0}{\sqrt{A(1-\delta)N}}}{\frac{1}{\sqrt{2\pi}} - \frac{E_0}{\sqrt{AN}}} \\ &\leq \frac{4E_0}{\sqrt{A}(1-\delta)}, \end{aligned}$$

for a sufficiently large N_1 and all $N \geq N_1$.

Case 3: $r > \rho_0 N$**.**

Again we write

$$\frac{\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\mu_{\Lambda,\varphi}(R = r)} = \sqrt{\frac{N}{N - |\Lambda_0|}} \frac{\sqrt{\sigma^2(N - |\Lambda_0|)}\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\sqrt{\sigma^2 N}\mu_{\Lambda,\varphi}(R = r)}$$

This time by the second part of Proposition 4.7 we have

$$\sqrt{\sigma^2(N - |\Lambda_0|)}\mu_{\Lambda,\varphi}(R_1 = r - r_0) \leq \frac{1}{\sqrt{2\pi}} + \frac{E_1}{\sqrt{N - |\Lambda_0|}} \leq \frac{1}{\sqrt{2\pi}} + \frac{E_1}{\sqrt{(1-\delta)N}}$$

and

$$\sqrt{\sigma^2 N}\mu_{\Lambda,\varphi}(R = r) \geq \frac{1}{\sqrt{2\pi}} - \frac{E_1}{\sqrt{N}} \geq \frac{1}{\sqrt{2\pi}} - \frac{E_1}{\sqrt{N}}$$

This implies that

$$\begin{aligned} \frac{\mu_{\Lambda,\varphi}(R_1 = r - r_0)}{\mu_{\Lambda,\varphi}(R = r)} &\leq \frac{1}{\sqrt{1-\delta}} \frac{\frac{1}{\sqrt{2\pi}} + \frac{E_1}{\sqrt{(1-\delta)N}}}{\frac{1}{\sqrt{2\pi}} - \frac{E_1}{\sqrt{N}}} \\ &\leq \frac{4}{\sqrt{1-\delta}}, \end{aligned}$$

for a sufficiently large N_2 and all $N \geq N_2$. To complete the proof we choose $n_0 = \max\{N_0, N_1, N_2\}$.

□

A direct consequence of this is the following result.

Corollary 4.10. *Under the conditions defined above, given a function f whose support is restricted to Λ_0, there exists a constant C and an n_0 such that*

(i) $\nu_{\Lambda,r}[f] \leq C\mu_{\varphi(\frac{r}{|\Lambda|})}[|f|]$

(ii) $|\nu_{\Lambda,r}[f] - \mu_{\varphi(\frac{r}{|\Lambda|})}[f]| \leq C\sqrt{\mu_{\varphi(\frac{r}{|\Lambda|})}[f;f]}$

uniformly for $|\Lambda| \geq n_0$ and all $r > 0$.

The first result follows directly from Corollary 4.9. The second result follows from the first by plugging in $f - \mu_\varphi(f)$ into the first and applying the Cauchy-Schwarz inequality.

If instead we consider a function which is local, with support of $\Lambda_0 \subset \Lambda$,we may apply the local central limit theorem in a similar manner as in the corollaries to obtain the following result.

Corollary 4.11. *Consider a function f whose support is again Λ_0. Fix $\rho_0 > 0$. In the formulae below let $\varphi = \varphi_\Lambda\left(\frac{r}{|\Lambda|}\right)$.*

(i) *There exists constants C and A and an n_0 such that*

$$\begin{aligned}&|\nu_{\Lambda,r}[f] - \mu_\varphi[f]|\\ &\le C\frac{|\Lambda_0|}{|\Lambda|}\left\{\frac{1}{\sigma^2_\Lambda(\varphi)}\mu_{\Lambda,\varphi}[|f-\mu_{\Lambda,\varphi}[f]|] + \frac{1}{\sigma_\Lambda(\varphi)}\sqrt{\mu_{\Lambda,\varphi}[f;f]}\right\}\end{aligned}$$

for any $|\Lambda| \ge n_0$ and all r such that $\frac{A}{|\Lambda|} \le \frac{r}{|\Lambda|} \le \rho_0$.

(ii) *There exists a constant C and an n_0 such that*

$$|\nu_{\Lambda,r}[f] - \mu_\varphi[f]| \le C\frac{|\Lambda_0|}{|\Lambda|}\sqrt{\mu_{\Lambda,\varphi}[f;f]}$$

uniformly for $|\Lambda| \ge n_0$ and all $\frac{r}{|\Lambda|} > \rho_0$.

We may choose n_0 to be the same in both of these cases.

Proof. Consider the second case, and denote by ξ the configuration η restricted to the subset Λ_0. We continue using notation from the previous result. We may write

$$\nu_{\Lambda,r}[f] - \mu_{\Lambda,\varphi}[f] \le \sum_\xi |f(\xi) - \mu_{\Lambda,\varphi}[f]|\,\mu_\varphi(\xi)\left\{\frac{\mu_\varphi[R_1 = r - r_0]}{\mu_\varphi[R=r]} - 1\right\}. \tag{4.7}$$

We therefore need to bound the difference inside the brackets to complete the proof. As before, we use the second part of the local limit theorem with $J = 3$.

$$\begin{aligned}&\frac{\mu_\varphi[R_1 = r-r_0]}{\mu_\varphi[R=r]} - 1\\ &\le\ C\{\sqrt{\sigma^2(N-|\Lambda_o|)}\mu_\varphi[R_1 = r-r_0] - \sqrt{\sigma^2 N}\mu_\varphi[R=r]\}\\ &\le\ C\left\{\frac{E_0}{N} + g_0(0) - g_0\left(\frac{r_0}{\sigma\sqrt{N-|\Lambda_0|}}\right) - \frac{1}{\sqrt{N}}g_1\left(\frac{r_0}{\sigma\sqrt{N-|\Lambda_0|}}\right)\right\}\\ &\le\ C\frac{C(E_0)|\Lambda_0|}{N}\left\{1 + \frac{r_0}{\sigma|\Lambda_0|} + \left(\frac{r_0}{\sigma|\Lambda_0|}\right)^2\right\}.\end{aligned}$$

We plug this estimate into (4.7) and apply Cauchy-Schwarz inequality. To finish we apply the bounds from the first part of Proposition 4.5 to get rid of the extra

moment terms. A similar argument proves the first case, using the first part of Theorem 4.7. □

Using the same approach, but setting $J = 4$ in the local limit theorem expansion, we obtain a further decomposition:

Corollary 4.12. *Consider a function $f = f(\eta_x)$ for a fixed $x \in \Lambda$. Fix $\rho_0 > 0$. In the formulae below let $\varphi = \varphi_\Lambda\left(\frac{r}{|\Lambda|}\right)$. There exists constants C and A and an n_0 such that*

$$\begin{aligned}&|\nu_{\Lambda,r}[f] - \mu_\varphi[f] - \frac{1}{2|\Lambda|}\{\mu_{\Lambda,\varphi}[f;\eta_x - \rho_x] + \mu_\varphi[f;(\eta_x-\rho_x)^2]\}|\\ &\leq C\frac{1}{|\Lambda|^{3/2}}\left\{\frac{1}{\sigma^2_\Lambda(\varphi)}\sqrt{\mu_{\Lambda,\varphi}[f;f]}\right\}\end{aligned}$$

for any $|\Lambda| \geq n_0$ and all r such that $\frac{A}{|\Lambda|} \leq \frac{r}{|\Lambda|} \leq \rho_0$.

5. Some Birth and Death Processes.

As we mentioned previously, in this section we establish spectral gap and logarithmic Sobolev inequalities for several birth and death chains which arise naturally in the study of the second half of (3.3).

5.1. **Some Spectral Gaps.** We begin by stating a result whose proof appears as Lemma 4.3 in [LSV].

Lemma 5.1. *Let Y_t be a birth and death process on $\{0, 1, \ldots, r\}$ with death rate $d(\cdot)$ and birth rate $b(\cdot)$. Assume that there exists a finite positive constant J_2 such that*

$$\sup_k |b(k+1) - b(k)| \leq J_2, \tag{5.1}$$

and that there exist finite constants $J_0 > 0$ and $J_1 > J_2$ such that

$$d(k) - d(j) \geq J_1(k-j) - J_0, \tag{5.2}$$

for all $k \geq j$. Then the spectral gap for this process is bounded below by a strictly positive constant λ depending on J_0, J_1, J_2 and $d^* = \min_{k\geq 1} d(k)$.

From the above Lemma follow the next two results.

Lemma 5.2. *Under assumptions (LG) and (M), there exists a constant $B_0 = B_0(a_1, a_2, k_0)$ such that*

$$\mu_{\Lambda,\varphi}[\phi;\phi] \leq B_0\ \mu_{\Lambda,\varphi}[c_x(\eta_x)\{\phi(\eta_x - 1) - \phi(\eta_x)\}^2]$$

for all functions $\phi : \mathbb{N} \mapsto \mathbb{R}$ with $\mu_{\Lambda,\varphi}[\phi^2] < \infty$.

Proof. A straightforward calculation using the properties of the marginal μ_φ shows that

$$\begin{aligned}&\mu_{\Lambda,\varphi}[c_x(\eta_x)\{\phi(\eta_x-1)-\phi(\eta_x)\}^2]\\&=\frac{1}{2}\mu_{\Lambda,\varphi}[c_x(\eta_x)\{\phi(\eta_x-1)-\phi(\eta_x)\}^2]+\frac{\varphi}{2}\mu_{\Lambda,\varphi}[\{\phi(\eta_x+1)-\phi(\eta_x)\}^2]\end{aligned}\tag{5.3}$$

This is the Dirichlet form corresponding to the birth and death chain with death rate $c_x(\cdot)$ and birth rate φ. Since the birth rate is constant, and the death rate satisfies assumption (5.2) with $J_1 = a_2/k_0$ and $J_0 = a_2 + a_1k_0$, the result follows. Also, since the death rate $c_x(k) \geq c_1k$ for all k we obtain a uniform lower bound d^*, which depends on a_1 and a_2. □

Lemma 5.3. *Under the uniform assumptions (LG) and (M), there exists a constant $B_0 = B_0(a_1, a_2, k_0)$ such that*

$$\nu_{\Lambda,r}[f;f] \leq B_1\nu_{\Lambda,r}[c_x(\eta_x)\{f(\eta_x-1)-f(\eta_x)\}^2]\tag{5.4}$$

for all functions $f : \mathbb{N} \mapsto \mathbb{R}$ with $\nu_{\Lambda,r}[f^2] < \infty$, and all Λ such that $|\Lambda| \geq 1$ and $r \geq 1$.

Proof. Note that the marginals $\nu_{\Lambda,r}$ satisfy the relationship

$$\nu_{\Lambda,r}(\ \cdot\ |\eta_x = k) = \nu_{\Lambda_x,r-k}(\ \cdot\ ),$$

where $\Lambda_x = \Lambda \setminus \{x\}$. Using this fact, a similar calculation to the one in (5.3) reveals that this corresponds to the birth and death chain with death rate $c_x(\cdot)$ and birth rate $b_x(k) = AV_{y\sim x}\nu_{\Lambda_x,r-k}[c_y(\eta_y)]$. As above, we need only show that the birth rate satisfies the necessary conditions. That is, we need to show that there exists a constant C such that for any Λ, r and site x,

$$|\nu_{\Lambda,r+1}[c_x(\eta_x)] - \nu_{\Lambda,r}[c_x(\eta_x)]| \leq C$$

We split this up into three cases. Let $\rho = \frac{r}{|\Lambda|}$ and fix $\rho_0 > 0$. From Proposition 4.11 we choose an n_0 and A.

We first assume that $\frac{A}{|\Lambda|} \leq \rho \leq \rho_0$. Choosing $\phi(\eta) = c_x(\eta_x)$ in the first part of Proposition 4.11 we have that $|\nu_{\Lambda,r}[c_x(\eta_x)] - \varphi(\rho)| \leq \frac{1}{|\Lambda|}C(\rho_0)$, for some constant C depending on ρ_0. From this it follows that

$$|\nu_{\Lambda,r+1}[c_x(\eta_x)] - \nu_{\Lambda,r}[c_x(\eta_x)]| \leq 2C(\rho_0)/|\Lambda|.$$

Next assume that $\rho > \rho_0$. Here we may write

$$\begin{aligned}&|\nu_{\Lambda,r+1}[c_x(\eta_x)] - \nu_{\Lambda,r}[c_x(\eta_x)]|\\&\leq \textstyle\sum_{k\geq 0}|c_x(k)-\varphi(\rho)|\mu_{\Lambda,\varphi}(k)\left\{\frac{\mu_{\Lambda,\varphi}(R=r)\mu_{\Lambda,\varphi}(R=r+1-k)-\mu_{\Lambda,\varphi}(R=r-k)\mu_{\Lambda,\varphi}(R=r+1)}{\mu_{\Lambda,\varphi}(R=r+1)\mu_{\Lambda,\varphi}(R=r)}\right\}.\end{aligned}$$

Using the first part of Proposition 4.7 with the expansion up to $J = 3$, we bound the term inside the brackets by $C\frac{k}{|\Lambda|}$, from which the desired bound follows.

To handle the last case, namely $r \geq A$ and $|\Lambda| \leq n_0$, we use the monotonicity results of Lemma 4.3. We fix B sufficiently large as in the requirement of the lemma,

and set $M = B|\Lambda|$. There exists a measure Q on $\mathcal{X} \times \mathcal{C}$ with marginals $\nu_{\Lambda,r}$ and $\nu_{\Lambda,r+M+1}$ such that $\nu_{\Lambda,r} \leq \nu_{\Lambda,r+M+1}$. We thus have

$$\begin{aligned}
&|\nu_{\Lambda,r+1}[c_x(\eta_x)] - \nu_{\Lambda,r}[c_x(\eta_x)]| \\
&\quad \leq \quad |\nu_{\Lambda,r+1}[c_x(\eta_x)] - \nu_{\Lambda,r+M+1}[c_x(\eta_x)]| + \quad |\nu_{\Lambda,r+M+1}[c_x(\eta_x)] - \nu_{\Lambda,r}[c_x(\eta_x)]|,
\end{aligned}$$

which is smaller than $2a_1$ by assumption (LG) on the rates. This last fact, together with a bound on the finitely many remaining rates for $k \leq A$ proves that the birth rates satisfy (5.1). Notice again that the lower bound d^* is uniform in the sites. $\square$

5.2. **A Logarithmic Sobolev Inequality.** In the remainder of this section we prove a logarithmic Sobolev inequality for yet another birth death process. Up to now we have considered processes formed by considering marginal dynamics on a single site of zero range. These single site marginals are used mainly in the proof of Theorem 2.3. As we explain in Section 8, the proof of the spectral gap is also an induction argument. The argument differs in that with each induction step we *add* one site to the set Λ. In the proof of the logarithmic Sobolev inequality we *double* the size of Λ at each induction step. Hence, we will need a logarithmic Sobolev inequality which acts on the number of particles moving between two subsets of Λ, and not on a single site. In the remainder of this section we *do* need to use the assumption (E).

Let $\gamma_1(r_1) = \nu_{\Lambda,R}(R_1 = r_1)$. Recall that R_1 is the random variable which counts the total number of particles in subset Λ_1. The function $\gamma_1(r_1)$ is a probability measure on $\{0, 1, \ldots, r\}$ that is reversible for the birth and death process with generator

$$\begin{aligned}
L^{bd}\psi(r_1) \quad &= \quad \left[\frac{\gamma_1(r_1+1)}{\gamma_1(r_1)} \wedge 1\right]\{\psi(r_1+1) - \psi(r_1)\} \\
&\quad + \left[\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)} \wedge 1\right]\{\psi(r_1-1) - \psi(r_1)\} \qquad (5.5)
\end{aligned}$$

and Dirichlet form

$$D^{bd}(\psi) = \sum_{r_1=1}^{r} [\gamma_1(r_1) \wedge \gamma_1(r_1-1)]\{\psi(r_1) - \psi(r_1-1)\}^2$$

The work of Miclo in [M1] allows us to check that this birth and death process satisfies a logarithmic Sobolev inequality of its own:

Proposition 5.4. *The birth death process defined through the generator (5.5) satisfies a logarithmic Sobolev inequality*

$$H(\psi|\gamma(r_1)) \leq CrD^{bd}(\sqrt{\psi})$$

for some constant $C > 0$, independent of the sites in Λ_1, for all $\psi \geq 0$.

We divide the proof into several steps. In [M1], necessary and sufficient conditions on the rates for birth and death processes are given so that a logarithmic Sobolev inequality holds. In [CMR] it is proved that the Miclo conditions are satisfied by

probability measures satisfying certain exponential bounds. Together these results imply the following statement:

Lemma 5.5. *For a birth and death process as described above, suppose that there exists a constant A_0 such that for any integer $r > 0$ we can find $\bar{r} \in \{0, \ldots, r\}$ such that $A_0^{-1}\bar{r} \le r - \bar{r} \le A_0\bar{r}$ and*

$$\frac{\gamma_1(r_1+1)}{\gamma_1(r_1)} \le e^{-\left(\frac{r_1-\bar{r}}{A_0\bar{r}}\right)} \quad \text{for } r_1 \in \{\bar{r}+1, \ldots, r\}, \tag{5.6}$$

$$\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)} \le e^{-\left(\frac{r_1-\bar{r}}{A_0\bar{r}}\right)} \quad \text{for } r_1 \in \{0, \ldots, \bar{r}-1\}, \tag{5.7}$$

$$\frac{1}{A_0\sqrt{\bar{r}}} e^{-\left(\frac{r_1-\bar{r}}{A_0\bar{r}}\right)^2} \le \gamma_1(r_1) \le \frac{A_0}{\sqrt{\bar{r}}} e^{-\left(\frac{r_1-\bar{r}}{A_0\bar{r}}\right)^2} \quad \text{for } r_1 \in \{0, \ldots, r\}. \tag{5.8}$$

Then, there exists a positive constant A_1 such that for any positive function ψ on $\{0, \ldots, r\}$,

$$H(\psi|\gamma_1(\cdot)) \le A_1 r \sum_{k=1}^{r} [\gamma_1(r) \wedge \gamma_1(r)][\psi(r) - \psi(r-1)]^2,$$

for any integer $r > 0$.

The proof of the above appears in [CMR] or [DPP1].

If we can show that our probabilities γ_1 satisfy (5.6) through (5.8) we will have proved Proposition 5.4. It turns out that these conditions are satisfied by a modified measure, $\gamma_1^{\varepsilon_0}$, which is equivalent to γ_1. By the standard comparison method the logarithmic Sobolev inequality then follows for γ_1. We proceed by defining a class of equivalent measures γ_1^{ε}, and then finding a particular value of ε, ε_0 so that (5.6) through (5.8) are satisfied.

We begin with a technical result. Assume that Λ is of size N.

Lemma 5.6.

$$0 < \inf_{\Lambda, r>0} \sigma_\Lambda\left(\frac{r}{N}\right)\sqrt{N}\mu_{\Lambda,\varphi_\Lambda(\frac{r}{N})}(R = r) \le \sup_{\Lambda, r>0} \sigma_\Lambda\left(\frac{r}{N}\right)\sqrt{N}\mu_{\Lambda,\varphi_\Lambda(\frac{r}{N})}(R = r) < \infty$$

Proof. This result follows for most cases directly from the local limit theorems 4.7 and 4.8. It remains to bound the only case not covered by these theorems: $N \le N_0$ and $r > \rho_0 N$, for some fixed N_0 and ρ_0. This last case is simply condition (E). $\square$

Remark 5.7. *As mentioned before, there are several simpler conditions such that condition (E) is satisfied. One we have mentioned already, namely, that there exists a K_0 such that*

$$c_x(k) = \theta k, \quad \forall\ x \text{ and } k \ge K_0.$$

To show that (E) is satisfied here is a straightforward, albeit lengthy, calculation using Stirling's formula. A similar argument also shows that (E) is satisfied if we assume

that there exists a large constant K_0, and two positive constants r_1 and r_2 such that for all $k \geq K_0$ the rate function c satisfies for all x

$$c_x(k) = \begin{cases} \theta_1 k & \textit{if } k \textit{ is odd,} \\ \theta_2 k & \textit{if } k \textit{ is even.} \end{cases}$$

In fact, any other similar pattern also works.

We next define the modified measures γ_1^ε with ε in $(0, 1/4)$. We will first show that these measures are equivalent to γ_1. We will then show that there is a special choice of ε such that the measures γ_1^ε satisfy all three conditions of Lemma 5.5. The modified measure γ_1^ε is defined as follows. Let $I_\varepsilon := [\varepsilon r, (1-\varepsilon) r] \cap \mathbb{Z}$, and set $\bar{r} = \lceil r/2 \rceil$. For r_1 in $\{0, \ldots, r\}$ we define the function

$$H(r_1) = \log \left\{ \frac{\mu_{\Lambda_1, \varphi_{\Lambda_1}(\frac{r_1}{|\Lambda_1|})}(R_1 = r_1) \mu_{\Lambda_2, \varphi_{\Lambda_2}(\frac{r-r_1}{|\Lambda_2|})}(R_2 = r - r_1)}{\mu_{\Lambda_1, \varphi_{\Lambda_1}(\frac{\bar{r}}{|\Lambda_1|})}(R_1 = r_1) \mu_{\Lambda_1, \varphi_{\Lambda_2}(\frac{\bar{r}}{|\Lambda_2|})}(R_2 = r - r_1)} \right\},$$

and use it to define the normalizing constant

$$Z = \frac{\sum_{k \in I^\varepsilon} e^{-H(k)}}{\sum_{k \in I^\varepsilon} \gamma_1(k)}.$$

We may now define the new measure

$$\gamma_1^\varepsilon(r_1) = \begin{cases} e^{-H(r_1)}/Z & \text{if } r_1 \in I_\varepsilon \\ \gamma_1(r_1) & \text{otherwise.} \end{cases} \tag{5.9}$$

We next use Lemma 5.6 to show that the two measures γ_1^ε and γ_1 are equivalent.

Lemma 5.8. *For any fixed $0 < \varepsilon < \frac{1}{4}$ there exists a positive constant C such that*

$$\frac{1}{C} \leq \frac{\gamma_1^\varepsilon(r_1)}{\gamma_1(r_1)} \leq C,$$

for all $r > 0$, Λ and $r_1 \in \{0, \ldots, r\}$.

Proof. We need only check bounds inside I_ε. Define

$$\pi(r_1) = \frac{\mu_{\Lambda_1, \varphi_{\Lambda_1}(\frac{r_1}{|\Lambda_1|})}(R_1 = r_1) \mu_{\Lambda_1, \varphi_{\Lambda_2}(\frac{r-r_1}{|\Lambda_2|})}(R_2 = r - r_1)}{\mu_{\Lambda, \varphi_\Lambda(\frac{r}{|\Lambda|})}(R = r)}.$$

As we may write

$$\frac{\gamma_1(r_1)}{\gamma_1^\varepsilon(r_1)} = \frac{\sum_{k \in I_\varepsilon} \gamma_1(k) \frac{\pi(r_1)}{\pi(k)}}{\gamma_1(k)}$$

to prove the result it is enough to bound the ratio $\frac{\pi(r_1)}{\pi(k)}$ uniformly as k and r_1 vary over I_ε. Define

$$B(n,m)^2 = \frac{\sigma^2_{\Lambda_1}(\frac{m}{|\Lambda_1|}) \sigma^2_{\Lambda_2}(\frac{r-m}{|\Lambda_2|})}{\sigma^2_{\Lambda_1}(\frac{n}{|\Lambda_1|}) \sigma^2_{\Lambda_2}(\frac{r-n}{|\Lambda_2|})}$$

By Lemma 5.6 we have that there exists a positive finite constant C_0 such that

$$\frac{1}{C_0}B(r_1,\bar r) \le \frac{\pi(r_1)}{\pi(k)} \le C_0 B(r_1,\bar r).$$

To finish notice the following

$$\frac{1}{C_1}4\epsilon(1-\epsilon) \le \frac{r_1(r-r_1)}{C_1\bar r(r-\bar r)} \le B^2(r_1,\bar r) \le \frac{C_1 r_1(r-r_1)}{\bar r(r-\bar r)} \le \frac{C_1}{4\epsilon(1-\epsilon)}.$$

The result follows. $\square$

Our next goal is to show that the measure $\gamma_1^\epsilon(\cdot)$ satisfies conditions (5.6) through (5.8). We first need the following.

Proposition 5.9.

$$C^{-1}\frac{r_1}{r-r_1+1} \le \frac{\gamma_1(r_1-1)}{\gamma_1(r_1)} \le C\frac{r_1}{r-r_1+1}.$$

Proof. We begin with the following identity:

$$\begin{aligned}
\mu_{\Lambda,\varphi}[R=r+1] &= \frac{1}{r+1}\sum_{x\in\Lambda}\mu_{\Lambda,\varphi}[\eta_x\cdot\mathbb{I}[R=r+1]] \\
&= \frac{1}{r+1}\sum_{x\in\Lambda}\mu_{\Lambda,\varphi}[\frac{\varphi\cdot(\eta_x+1)}{c_x(\eta_x+1)}\mathbb{I}[R=r+1]].
\end{aligned}$$

Using the uniform bounds on $\frac{c_x(k)}{k}$ we obtain for some $B>0$

$$\frac{\varphi|\Lambda|\mu_{\Lambda,\varphi}[R=r]}{B(r+1)} \le \mu_{\Lambda,\varphi}[R=r+1] \le \frac{B\varphi|\Lambda|\mu_{\Lambda,\varphi}[R=r]}{r+1}.$$

Together with

$$\frac{\gamma_i(r_i-1)}{\gamma_i(r_i)} = \frac{\mu_{\Lambda_i,\varphi}[R_i=r_i-1]\mu_{\Lambda_j,\varphi}[R_j=r-r_i+1]}{\mu_{\Lambda_i,\varphi}[R_i=r_i]\mu_{\Lambda_j,\varphi}[R_j=r-r_j]},$$

for $i\neq j$. This implies the result. $\square$

Proposition 5.9 implies that for all ε smaller than $1/3$ we have that

$$\begin{aligned}
\frac{\gamma_1^\varepsilon(r_1+1)}{\gamma_1^\varepsilon(r_1)} &\le \tfrac{1}{2} && \text{for } r_1\in[(1-\varepsilon)r, r-1] \\
\frac{\gamma_1^\varepsilon(r_1-1)}{\gamma_1^\varepsilon(r_1)} &\le \tfrac{1}{2} && \text{for } r_1\in[1,\varepsilon r]
\end{aligned}$$

for any $r>0$, and any Λ with $|\Lambda|\ge 2$. We next prove the following lemma, which together with the above statement implies that conditions (5.6) and (5.7) are satisfied.

Lemma 5.10. *For any $\varepsilon \in (0,1/4)$ there exists a positive constant A_0 such that for any $r > 0$*

$$\frac{r_1-\bar{r}}{A_0\bar{r}} \leq H(r_1+1)-H(r_1) \leq A_0\frac{r_1-\bar{r}}{\bar{r}} \qquad \text{for any } r_1 \text{ in } I_\varepsilon \text{ and } r_1 > \bar{r}$$
$$\frac{r_1-\bar{r}}{A_0\bar{r}} \leq H(r_1-1)-H(r_1) \leq A_0\frac{r_1-\bar{r}}{\bar{r}} \qquad \text{for any } r_1 \text{ in } I_\varepsilon \text{ and } r_1 < \bar{r}$$

Proof. A careful calculation reveals

$$\partial_z H(z) = \log\varphi_{\Lambda_1}(z/m) - \log\varphi_{\Lambda_2}(r-z/m),$$

where $m = N/2 = |\Lambda_1|$. We differentiate again to obtain

$$\partial_z^2 H(z) = \frac{1}{\sigma^2_{\Lambda_1}(z/m)m} - \frac{1}{\sigma^2_{\Lambda_2}((r-z)/m)m}.$$

Using the bounds from Proposition 4.4 we have that there exists a positive constant B so that

$$\frac{1}{B}\left\{\frac{1}{z}+\frac{1}{r-z}\right\} \leq \partial_z^2 H(z) \leq B\left\{\frac{1}{z}+\frac{1}{r-z}\right\}$$

for all z in $[\bar{r},(1-\varepsilon)r]$, and thus we can find a constant $B_2 = B_2(\varepsilon)$ so that

$$\frac{1}{B_2}\left\{\frac{1}{\bar{r}}\right\} \leq \partial_z^2 H(z) \leq B_2\left\{\frac{1}{\bar{r}}\right\}.$$

Integrating once from $\bar{r}$ to z and then again from z to $z+1$ we obtain the first part of the inequality. We repeat the argument to obtain the other direction. $\square$

It remains to prove the third condition.

Lemma 5.11. *There exists $\varepsilon_o \in (0,1/4)$ and a positive constant A_0 such that*

$$\frac{1}{A_0\sqrt{\bar{r}}}e^{-\left(\frac{r-\bar{r}}{A_0\bar{r}}\right)} \leq \gamma_1^{\varepsilon_o}(r) \leq \frac{A_0}{\sqrt{\bar{r}}}e^{-\left(\frac{r-\bar{r}}{A_0\bar{r}}\right)}$$

Proof. We split the proof into several steps. We will also make use of the fact that the measures satisfy conditions (5.6) and (5.7).

Step 1. We write the arguments below for $r_1 > \bar{r}$; the argument in the opposite direction is the same.

$$\begin{aligned}
\log\frac{\gamma_1^\varepsilon(r_1)}{\gamma_1^\varepsilon(\bar{r})} &= \sum_{k=\bar{r}}^{r_1-1}\log\frac{\gamma_1^\varepsilon(k+1)}{\gamma_1^\varepsilon(k)} \\
&\leq -\sum_{k=\bar{r}}^{r_1-1}\frac{k-\bar{r}}{A_0\bar{r}} \\
&\leq -\frac{(r_1-\bar{r})^2}{2A_0\bar{r}} + \frac{1}{2A_0}
\end{aligned}$$

for any r_1. This implies that there exists an $A_1 > 0$ so that

$$\frac{\gamma_1^\varepsilon(r_1)}{\gamma_1^\varepsilon(\bar{r})} \leq A_1 e^{-\frac{(r_1-\bar{r})^2}{A_1\bar{r}}}.$$

Step 2. We repeat a similar argument using Lemma 5.10 to obtain

$$\frac{\gamma_1^\varepsilon(r_1)}{\gamma_1^\varepsilon(\bar{r})} \geq A_2 e^{-\frac{(r_1-\bar{r})^2}{A_2\bar{r}}},$$

for some $A_2 > 0$ and for any r_1 in I_ε. We are restricted to I_ε as that is where we obtain the necessary lower bound.

Step 3. We now extend the above to r_1 outside of I_ε. We work in one direction first, assuming that $r_1 > (1-\epsilon)r$. Let $\tilde{r} = \lfloor (1-\epsilon)r \rfloor$,

$$\begin{aligned}
\frac{\gamma_1^\varepsilon(r_1)}{\gamma_1^\varepsilon(\bar{r})} &= \frac{\gamma_1(r_1)}{\gamma_1(\tilde{r}+1)} \frac{\gamma_1(\tilde{r}+1)}{\gamma_1^\varepsilon(\tilde{r})} \frac{\gamma_1^\varepsilon(\tilde{r})}{\gamma_1^\varepsilon(\bar{r})} \\
&\geq C^{-1}(\varepsilon) \frac{\gamma_1(r_1)}{\gamma_1(\tilde{r})} \frac{\gamma_1^\varepsilon(\tilde{r})}{\gamma_1^\varepsilon(\bar{r})},
\end{aligned}$$

by Lemma 5.8. For the first fraction we have Proposition 5.9 which gives

$$\begin{aligned}
\log \frac{\gamma_1(r_1)}{\gamma_1(\tilde{r})} &\geq \sum_{k=\tilde{r}}^{r_1-1} \log \frac{r-k}{C(k+1)} \\
&\geq -r\log C - \frac{r}{\epsilon}.
\end{aligned}$$

Combining this with the results of step 2, we have that

$$\frac{\gamma_1^\varepsilon(r_1)}{\gamma_1^\varepsilon(\bar{r})} \geq \frac{A_2}{C} e^{-\frac{(\tilde{r}-\bar{r})^2}{A_2\bar{r}}} e^{-\log Cr - r/\varepsilon}.$$

We select an $\varepsilon = \varepsilon_0$ sufficiently small so that we obtain a positive constant C_2 so that

$$\begin{aligned}
\frac{\gamma_1^{\varepsilon_0}(r_1)}{\gamma_1^{\varepsilon_0}(\bar{r})} &\geq \frac{A_2}{C} e^{-C_2 r} \\
&\geq \frac{A_2}{C} e^{-64C\frac{(r_1-\bar{r})^2}{\bar{r}}}.
\end{aligned}$$

Repeating the argument in the opposite direction, we obtain that there exists a positive A_3 such that for any r_1

$$\frac{1}{A_3} e^{-A_3\frac{(r_1-\bar{r})^2}{\bar{r}}} \leq \frac{\gamma_1^{\varepsilon_o}(r_1)}{\gamma_1^{\varepsilon_o}(\bar{r})} \leq A_3 e^{-A_3\frac{(r_1-\bar{r})^2}{\bar{r}}}.$$

Note that if repeating the argument produces a smaller ε_0, we simply take the smaller of the two.

Step 4. We next sum the above fractions in r_1 to obtain the following bound

$$\frac{1}{A_4\sqrt{\bar{r}}} \leq \gamma_1^{\varepsilon_0}(\bar{r}) \leq \frac{A_4}{\sqrt{\bar{r}}}$$

for some constant A_4. This together with the previous bound implies condition (5.8). □

6. Independence of the number of particles

We now have the tools necessary to proceed with the first part of the proof of the main result, Theorem 2.1. In this section we finish the argument given in the outline given in Section 3, which allows us to establish that the logarithmic constant is independent of r, the total number of particles. There are two things we need to do in order to establish this result. We first need to establish the initial induction step from line (3.7). Second, we need to obtain the bound in (3.4):

$$H(\nu_{\Lambda,R}(f|R_1=r_1)|\nu_{\Lambda,r}) \leq C(N)(1+\kappa(N,r))D_{\Lambda,r}(\sqrt{f}) + C(N)\nu_{\Lambda,r}[f],$$

where $C(N)$ is a large positive constant depending on N. We begin with the latter, and establish the initial induction result in Proposition 6.6. The proof of (3.4) is computationally intensive; hence, for ease of reading, we split it up into several steps. In step 1, we reduce the problem to calculating bounds on two covariances. These estimates are provided in steps 2 and 3. In step 4 we combine these bounds to obtain the above result. All of the work involved is essentially identical to that for the homogeneous problem. The only differences lie in that many functions we now estimate depend on the site, and hence we need to check that all the bounds hold uniformly. However, as most of these bounds rely on single site estimates, the uniform bounds (LG) and (M) on the rates are sufficient for the results to hold.

Step 1: Initial Calculations. In Section 5 we defined a specific birth and death process, and showed that this process satisfies a logarithmic Sobolev inequality. We begin by applying this result to the term $H(\nu_{\Lambda,R}[f|R_1=r_1]|\nu_{\Lambda,r})$.

$$\begin{aligned} H(\nu_{\Lambda,R}[f|R_1]|\nu_{\Lambda,r}) &= H(\psi(R_1)|\nu_{\Lambda,r}) \\ &= H(\psi|\gamma_1(\cdot)) \\ &\leq Cr\sum_{r_1=1}^{r}[\gamma_1(r_1)\wedge\gamma_1(r_1-1)](\sqrt{\psi(r_1)}-\sqrt{\psi(r_1-1)})^2. \end{aligned}$$

Recall that $\gamma_1(k) = \nu_{\Lambda,r}(R_1 = k)$. We continue

$$\begin{aligned}
&H(\nu_{\Lambda,R}[f|R_1]|\nu_{\Lambda,r}) \\
&\leq \; Cr\sum_{r_1=1}^{r} \gamma_1(r_1)\wedge\gamma_1(r_1-1)\left(\sqrt{\nu_{\Lambda,r}(f|R_1=r_1)}-\sqrt{\nu_{\Lambda,r}(f|R_1=r_1-1)}\right)^2 \\
&\leq \; Cr\sum_{r_1=1}^{r}\Bigg\{\left[\frac{\gamma_1(r_1)\wedge\gamma_1(r_1-1)}{\nu_{\Lambda,r}(f|R_1=r_1)\vee\nu_{\Lambda,r}(f|R_1=r_1-1)}\right] \\
&\qquad\qquad \times(\nu_{\Lambda,r}(f|R_1=r_1)-\nu_{\Lambda,r}(f|R_1=r_1-1))^2\Bigg\} \qquad (6.1)
\end{aligned}$$

where the inequality $(\sqrt{a}-\sqrt{b})^2 \leq \frac{(a-b)^2}{a\vee b}$, for a and b positive, was used in the above.

Proposition 6.1. *For every f and $r_1 = 1,2,\dots,r$ we have*

$$\begin{aligned}
&\nu_{\Lambda,r}(f \mid R_1=r_1)-\nu_{\Lambda,r}(f \mid R_1=r_1-1) \\
&= \frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\frac{1}{r_1N}\left[\nu_{\Lambda,r}\left(\sum_{x\in\Lambda_1,y\in\Lambda_2}h_x(\eta_x)c_y(\eta_y)\nabla_{y,x}f\,\middle|\,R_1=r_1-1\right)\right. \\
&\qquad \left.+\nu_{\Lambda,r}\left(f;\sum_{x\in\Lambda_1,y\in\Lambda_2}h_x(\eta_x)c_y(\eta_y)\,\middle|\,R_1=r_1-1\right)\right],
\end{aligned}$$

where $h_x(k) = \frac{k+1}{c_x(k+1)}$. Moreover, by exchanging the roles of Λ_1 and Λ_2 we obtain for $r_1 = 0,1,2,\dots,r-1$ (equivalently, $r_2 = 1,2,\dots,r$ where $r_2 = r - r_1$), where $\gamma_2(r_2) = \nu_{\Lambda,r}(R_2 = r_2)$

$$\begin{aligned}
&\nu_{\Lambda,r}(f \mid R_1=r_1)-\nu_{\Lambda,r}(f \mid R_1=r_1-1) \\
&= \frac{\gamma_2(r-r_1)}{\gamma_2(r-r_1+1)}\frac{1}{(r-r_1+1)N}\left[\nu_{\Lambda,r}\left(\sum_{x\in\Lambda_1,y\in\Lambda_2}h_y(\eta_y)c_x(\eta_x)\nabla_{x,y}f\,\middle|\,R_1=r_1\right)\right. \\
&\qquad \left.+\nu_{\Lambda,r}\left(f;\sum_{x\in\Lambda_1,y\in\Lambda_2}h_y(\eta_y)c_x(\eta_x)\,\middle|\,R_1=r_1\right)\right].
\end{aligned}$$

We define

$$\begin{aligned}
A_1(r_1) &= \nu_{\Lambda,r}\left(\sum_{x\in\Lambda_1,y\in\Lambda_2}h_x(\eta_x)c_y(\eta_y)\nabla_{y,x}f\,\middle|\,R_1=r_1\right), \\
B_1(r_1) &= \nu_{\Lambda,r}\left(f;\sum_{x\in\Lambda_1,y\in\Lambda_2}h_x(\eta_x)c_y(\eta_y)\,\middle|\,R_1=r_1\right),
\end{aligned}$$

and

$$A(r_1) = \begin{cases} \frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\frac{1}{r_1 N}\ A_1(r_1-1) & \text{for } r_1 > \frac{r}{2} \\ \frac{\gamma_2(r-r_1)}{\gamma_2(r-r_1+1)}\frac{1}{(r-r_1+1)N}\ A_1(r_1) & \text{for } r_1 \le \frac{r}{2} \end{cases} \tag{6.2}$$

$$B(r_1) = \begin{cases} \frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\frac{1}{r_1 N}\ B_1(r_1-1) & \text{for } r_1 > \frac{r}{2} \\ \frac{\gamma_2(r-r_1)}{\gamma_2(r-r_1+1)}\frac{1}{(r-r_1+1)N}\ B_1(r_1) & \text{for } r_1 \le \frac{r}{2}. \end{cases} \tag{6.3}$$

We combine both representations above to obtain

$$\nu_{\Lambda,r}(f \mid R_1 = r_1) - \nu_{\Lambda,r}(f \mid R_1 = r_1 - 1) = A(r_1) + B(r_1),$$

Our next steps will be to obtain bounds on the terms A and B.

Proof of Proposition 6.1. The proof is based on the following two calculations. For $y \in \Lambda_2$,

$$\begin{aligned} &\nu_{\Lambda,r}[f\mathbb{I}(\eta_x > 0)|R_1 = r_1] \\ &\quad = \begin{cases} \frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\nu_{\Lambda,r}\left[\frac{c_y(\eta_y)}{c_x(\eta_x+1)}f(\eta^{y,x})|R_1 = r_1 - 1\right] & \text{if } x \in \Lambda_1 \\ \nu_{\Lambda,r}\left[\frac{c_y(\eta_y)}{c_x(\eta_x+1)}f(\eta^{y,x})|R_1 = r_1\right] & \text{if } x \in \Lambda_2 \end{cases}. \end{aligned}$$

Next, write

$$\nu_{\Lambda,r}[f|R_1 = r_1] = -\frac{1}{r_1}\sum_{x\in\Lambda_1}\nu_{\Lambda,r}[\eta_x \cdot \nabla^{x,y} f|R_1 = r_1] + \frac{1}{r_1}\sum_{x\in\Lambda_1}\nu_{\Lambda,r}[\eta_x f(\eta^{x,y})|R_1 = r_1],$$

and plug in the above calculation, along with $h_x(\eta_x) = \frac{\eta_x+1}{c_x(\eta_x+1)}$ to obtain

$$\begin{aligned} &\nu_{\Lambda,r}[f|R_1 = r_1] \\ &\quad = \frac{\gamma_1(r_1-1)}{\gamma_1(r_1)r_1}\left(\nu_{\Lambda,r}\left[c_y(\eta_y)\sum_{x\in\Lambda_1} h_x(\eta_x)\cdot\nabla^{y,x} f\,\middle|\, R_1 = r_1 - 1\right]\right. \\ &\qquad\qquad + \left.\nu_{\Lambda,r}\left[c_y(\eta_y)\sum_{x\in\Lambda_1} h_x(\eta_x) f\,\middle|\, R_1 = r_1 - 1\right]\right). \end{aligned}$$

Setting $f = 1$ in the above formula we obtain that

$$\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)r_1}\nu_{\Lambda,r}\left[c_y(\eta_y)\sum_{x\in\Lambda_1} h_x(\eta_x)\,\middle|\, R_1 = r_1 - 1\right] = 1,$$

and hence we have

$$\begin{aligned}&\nu_{\Lambda,r}[f|R_1=r_1-1]\\ &= \frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\left(\nu_{\Lambda,r}\left[c_y(\eta_y)\sum_{x\in\Lambda_1}h_x(\eta_x)f\,\middle|\, R_1=r_1-1\right]\right.\\ &\qquad - \left.\nu_{\Lambda,r}\left[c_y(\eta_y)\sum_{x\in\Lambda_1}h_x(\eta_x);f\,\middle|\, R_1=r_1-1\right]\right).\end{aligned}$$

□

Step 2: bounds on A. Suppose $\Lambda=\Lambda_1\cup\Lambda_2$. For any $i=1,2$ define $\gamma_i(r_i)=\nu_{\Lambda,r}[R_i=r_i]$. Notice that by symmetry Proposition 5.9 applies also to γ_2.

Proposition 6.2. *Recall the definition of A from (6.2). There exists a constant $C>0$ such that*

$$\begin{aligned}A^2(r_1)\le C\frac{N^2}{r}\ &\nu_{\Lambda,r}(f\mid R_1=r_1)\vee\nu_{\Lambda,r}(f\mid R_1=r_1-1)\\ &\times\left[\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}D_{\nu_{\Lambda,r}(\cdot|R_1=r_1)}(\sqrt{f})+D_{\nu_{\Lambda,r}(\cdot|R_1=r_1)}(\sqrt{f})\right].\end{aligned}$$

Proof. We work out the case where $r_1>r/2$, as the argument in the other direction is identical. Because $h_x\le c_2$ and $\nabla_{y,x}f=\nabla_{y,x}\sqrt{f}\,[\sqrt{f}(\eta^{y,x})+\sqrt{f}(\eta)]$, we may use the Cauchy-Schwarz inequality to bound $A^2(r_1)$ by

$$\begin{aligned}&c_2^2\left[\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\frac{1}{r_1N}\right]^2\nu_{\Lambda,r}\left(\sum_{x\in\Lambda_1,y\in\Lambda_2}c_y(\eta_y)(\nabla_{y,x}\sqrt{f})^2\mid R_1=r_1-1\right)\\ &\qquad\times\nu_{\Lambda,r}\left(\sum_{x\in\Lambda_1,y\in\Lambda_2}c_y(\eta_y)(f(\eta^{y,x})+f(\eta))\mid R_1=r_1-1\right)\end{aligned}$$

We next change measure to move from $f(\eta^{y,x})$ back to f with $x\in\Lambda_1$ and $y\in\Lambda_2$ using

$$\nu_{\Lambda,r}[c_y(\eta_y)f(\eta^{y,x})|R_1=r_1-1]=\frac{\gamma_1(r_1)}{\gamma_1(r_1-1)}\nu_{\Lambda,r}[c_x(\eta_x)f(\eta)|R_1=r_1]$$

which follows from the detailed balance condition (2.2). This gives us a new bound of

$$\begin{aligned}&c_2^2\left[\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\frac{1}{r_1N}\right]^2\nu_{\Lambda,r}\left(\sum_{x\in\Lambda_1,y\in\Lambda_2}c_y(\eta_y)(\nabla_{y,x}\sqrt{f})^2\mid R_1=r_1-1\right)\\ &\times\sum_{x\in\Lambda_1,y\in\Lambda_2}\left\{\frac{\gamma_1(r_1)}{\gamma_1(r_1-1)}\nu_{\Lambda,r}\left(c_x(\eta_y)f(\eta)\mid R_1=r_1\right)+\nu_{\Lambda,r}\left(c_y(\eta_y)f(\eta)\mid R_1=r_1-1\right)\right\}.\end{aligned}$$

We bound $\sum_{x\in\Lambda_1,y\in\Lambda_2} c_y(\eta_y)$ with $c_2(r-r_1+1)N$, $\sum_{x\in\Lambda_1,y\in\Lambda_2} c_x(\eta_x)$ with $c_2 r_1 N$, and use Proposition 5.9 to obtain the new bound for $A^2(r_1)$

$$c_2^2\left[\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\frac{1}{r_1N}\right]\nu_{\Lambda,r}\left(\sum_{x\in\Lambda_1,y\in\Lambda_2} c_y(\eta_y)(\nabla_{y,x}\sqrt{f})^2 \mid R_1=r_1-1\right) \tag{6.4}$$
$$\times\left\{c_2\,C\nu_{\Lambda,r}[f|R_1=r_1]\vee\nu_{\Lambda,r}[f|R_1=r_1-1]\right\}. \tag{6.5}$$

We next bound $(\nabla_{y,x}\sqrt{f})^2$ with

$$C'N\sum_{e_z}(\sqrt{f(\eta^{x,e_{z+1}})}-\sqrt{f(\eta^{x,e_z})})^2,$$

for some positive constant C', where the sum is over e_z: sites which form a path from y to x. We use this along with repeated change of measure to bound (6.4) as before.

$$\begin{aligned}
&\left[\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\frac{1}{r_1N}\right]\nu_{\Lambda,r}\left[\sum_{x\in\Lambda_1,y\in\Lambda_2} c_y(\eta_y)(\nabla_{y,x}\sqrt{f})^2\middle| R_1=r_1-1\right]\\
&\le \left[C'\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\frac{1}{r_1}\right]\sum_{x\in\Lambda_1,y\in\Lambda_2}\left\{\sum_{e_z\in\Lambda_1}\nu_{\Lambda,r}\left[c_{e_z}(\eta_{e_z})(\nabla^{e_z,e_{z+1}}f)^2\middle| R_1=r_1-1\right]\right.\\
&\qquad\left.+\frac{\gamma_1(r_1)}{\gamma_1(r_1-1)}\sum_{e_z\in\Lambda_2}\nu_{\Lambda,r}\left[c_{e_z}(\eta_{e_z})(\nabla^{e_z,e_{z+1}}f)^2\middle| R_1=r_1\right]\right\}\\
&\le \frac{2c_2N^2}{r_1}\sum_{x\sim y}\left\{\frac{\gamma_1(r_1-1)}{\gamma_1(r_1)}\nu_{\Lambda,r}[c_x(\eta_x)(\nabla^{x,y}f)^2 \mid R_1=r_1-1]\right.\\
&\qquad\left.+\nu_{\Lambda,r}[c_x(\eta_x)(\nabla^{x,y}f)^2 \mid R_1=r_1]\right\}
\end{aligned}$$

This last bound together with (6.5) completes the proof. □

Step 3: bounds on B.

Proposition 6.3. *If $r_1\le r/2$ then*

$$\begin{aligned}
B^2(r_1) &\le C(N)\frac{\gamma_2^2(r_2)}{\gamma_2^2(r_2+1)}\nu_{\Lambda,r}[f|R_1=r_1]\left[\frac{r_1}{(r_2+1)^2}\nu_{\Lambda_2,r-r_1}[H(f|\nu_{\Lambda_1,r_1})]\right.\\
&\quad\left.+\frac{r_1^2}{(r_2+1)^3}\left\{\nu_{\Lambda,r}[f|R_1=r_1]+\nu_{\Lambda_1,r_1}[H(f|\nu_{\Lambda_2,r-r_1})]\right\}\right].
\end{aligned} \tag{6.6}$$

If $r_1 > r/2$ *then*

$$\begin{aligned} B^2(r_1) &\leq C(N)\frac{\gamma_1^2(r_1-1)}{\gamma_1^2(r_1)}\nu_{\Lambda,r}[f|R_1=r_1-1]\left[\frac{r_2+1}{r_1^2}\nu_{\Lambda_2,r-r_1+1}[H(f|\nu_{\Lambda_1,r_1-1})]\right. \\ &\quad \left. + \frac{r_1^2}{(r_2+1)^3}\Big\{\nu_{\Lambda,r}[f|R_1=r_1-1]+\nu_{\Lambda_1,r_1-1}[H(f|\nu_{\Lambda_2,r-r_1+1})]\Big\}\right]. \end{aligned}$$

We will prove here the case where $r_1 \leq r/2$. Recall that in this instance $B(r_1)$ is equal to

$$\frac{\gamma_2(r-r_1)}{\gamma_2(r-r_1+1)}\frac{1}{(r-r_1+1)N}\nu_{\Lambda,r}\left[f;\sum_{x\in\Lambda_1,y\in\Lambda_2}h_y(\eta_y)c_x(\eta_x)\middle| R_1=r_1\right].$$

We use the fact that $\nu_{\Lambda,r}(\cdot|R_1=r_1)=\nu_{\Lambda_1,r_1}\otimes\nu_{\Lambda_2,r-r_1}$ and that $\sum_x c_x$ and $\sum_y h_y$ act on η_{Λ_1} and η_{Λ_1} respectively. Hence,

$$\begin{aligned} &\nu_{\Lambda,r}\left[f;\sum_{x\in\Lambda_1,y\in\Lambda_2}h_y(\eta_y)c_x(\eta_x)\middle| R_1=r_1\right] \\ &\quad = \nu_{\Lambda_2,r-r_1}\left[\sum_{y\in\Lambda_2}h_y(\eta_y)\cdot\nu_{\Lambda_1,r_1}\left[f;\sum_{x\in\Lambda_1}c_x(\eta_x)\right]\right] \\ &\qquad + \nu_{\Lambda_2,r-r_1}\left[\nu_{\Lambda_1,r_1}[f];\sum_{y\in\Lambda_2}h_y(\eta_y)\right]\nu_{\Lambda_1,r_1}\left[\sum_{x\in\Lambda_1}c_x(\eta_x)\right]. \end{aligned}$$

Using the consequences of assumptions (LG) and (M) we thus get the following simple bound

$$\begin{aligned} B^2(r_1) \leq 2\frac{\gamma_2^2(r-r_1)}{\gamma_2^2(r-r_1+1)}&\left\{\frac{1}{c_1^2(r_2+1)^2}\nu_{\Lambda_1,r_1}\left[\nu_{\Lambda_2,r-r_1}[f];\sum_{x\in\Lambda_1}c_x(\eta_x)\right]^2\right. \\ &\left.+\frac{r_1^2c_2^2}{N^2(r_2+1)^2}\nu_{\Lambda_2,r-r_1}\left[\nu_{\Lambda_1,r_1}[f];\sum_{y\in\Lambda_2}h_y(\eta_y)\right]^2\right\}. \end{aligned} \tag{6.7}$$

To obtain bounds on the two remaining covariances we will make use of the entropy inequality

$$|\mu(f;g)| \leq \frac{\mu(f)}{s}\log(\mu(e^{s(g-\mu(g))})\vee\mu(e^{-s(g-\mu(g))}))+\frac{1}{s}H(f|\mu), \tag{6.8}$$

valid for any $s>0$, and we will optimize over s after obtaining bounds on $\mu(e^{s(g-\mu(g))})$ for μ and g of interest. In the proof we will make use of the quantity $r_x=\nu_{\Lambda,r}(\eta_x)$ and of the spectral gap from Theorem 2.3 .

Proposition 6.4. *For a subset Λ' of size $|\Lambda| = N'$ there exists a constant $C(N')$ such that for every $N' > 0$ and $t \in [-1, 1]$, the following hold*

$$\log\ \nu_{\Lambda',r'}\left[e^{t(c_x(\eta_x)-\nu_{\Lambda,r}[c_x(\eta_x)])}\right] \leq C(N')\ r'\ t^2$$
$$\nu_{\Lambda',r'}\left[e^{t\cdot r\cdot\{h_x(\eta_x)-\nu_{\Lambda,r}[h_x(\eta_x)]\}}\right] \leq C(N')e^{C(N')\{r't^2+\sqrt{r'}|t|\}}.$$

Proof. To simplify notation slightly we will denote Λ', N' and r' simply as Λ, N and r. We start with the first inequality. Notice that if r is bounded then a simple Taylor series expansion proves the bound. We may hence assume that r is larger than any finite constant we need. We begin with with the identity

$$\nu_{\Lambda,r}[c_x(\eta_x)f(\eta)] = \nu_{\Lambda,r}[c_x(\eta_x)]\nu_{N,r-1}[f(\eta+\delta_x)]. \tag{6.9}$$

Define $h(t) = \nu_{\Lambda,r}[e^{tc_x(\eta_x)}]$. It follows that

$$\begin{aligned} th'(t) - h(t)\log h(t) &= t\nu_{\Lambda,r}[c_x(\eta_x)e^{tc_x(\eta_x)}] - \nu_{\Lambda,r}[e^{tc_x(\eta_x)}]\log \nu_{\Lambda,r}[e^{tc_x(\eta_x)}] \\ &\leq t\nu_{\Lambda,r}[c_x(\eta_x)e^{tc_x(\eta_x)}] - t\nu_{\Lambda,r}[c_x(\eta_x)]\nu_{\Lambda,r}[e^{tc_x(\eta_x)}] \\ &= t\nu_{\Lambda,r}[c_x(\eta_x)]\left\{\nu_{N,r-1}[e^{tc_x(\eta_x+1)}] - \nu_{\Lambda,r}[e^{tc_x(\eta_x)}]\right\} \\ &\leq c_2 tr_x\left\{\nu_{N,r-1}[e^{tc_x(\eta_x+1)}] - \nu_{\Lambda,r}[e^{tc_x(\eta_x)}]\right\}. \end{aligned}$$

We next bound $\nu_{N,r-1}[e^{tc_x(\eta_x+1)}] - \nu_{\Lambda,r}[e^{tc_x(\eta_x)}]$. We split this bound into two parts. By the inequality $|e^x - e^y| \leq |x-y|e^{|x-y|}e^x$ we have

$$|\nu_{N,r-1}[e^{tc_x(\eta_x+1)}] - \nu_{N,r-1}[e^{tc_x(\eta_x)}]| \leq a_1te^{a_1t}\nu_{N,r-1}[e^{tc_x(\eta_x+1)}].$$

Next consider

$$\begin{aligned} &\left|\nu_{N,r-1}[e^{tc_x(\eta_x)}] - \nu_{\Lambda,r}[e^{tc_x(\eta_x)}]\right| \\ &\quad\leq \left|\nu_{N,r-1}[e^{tc_x(\eta_x)}] - \nu_{N,r-M}[e^{tc_x(\eta_x)}]\right| + \left|\nu_{\Lambda,r}[e^{tc_x(\eta_x)}] - \nu_{N,r-M}[e^{tc_x(\eta_x)}]\right| \end{aligned} \tag{6.10}$$

The two pieces are now dealt with in the same way, by Lemma 4.3 as long as M is large enough, there exists a coupling measure Q on $\{\eta, \xi\}$ such that the marginal of η is $\nu_{\Lambda,r}$, the marginal of ξ is $\nu_{N,r-M}$, and Q is concentrated on the configurations such that $\xi \leq \eta$. Hence we have

$$\begin{aligned} &|\nu_{\Lambda,r}[e^{tc_x(\eta_x)}] - \nu_{N,r-M}[e^{tc_x(\eta_x)}]| \\ &\quad= |Q[e^{tc_x(\eta_x)} - e^{tc_x(\xi_x)}]| \\ &\quad\leq tC'\nu_{\Lambda,r}[e^{tc_x(\eta_x)}], \end{aligned}$$

where C' is some constant depending on M. For the second term in (6.10) we obtain a bound of $tC'\nu_{N,r-1}[e^{tc_x(\eta_x)}]$. We next replace $\nu_{N,r-1}[e^{tc_x(\eta_x)}]$ with $\nu_{\Lambda,r}[e^{tc_x(\eta_x)}]$.

$$\begin{aligned}\nu_{N,r-1}[e^{tc_x(\eta_x)}] &\leq e^{c_2 t}\nu_{N,r-1}[e^{tc_x(\eta_x+1)}] \\ &= \frac{e^{c_2 t}}{\nu_{N,r-1}[c_x(\eta_x)]}\nu_{\Lambda,r}[c_x(\eta_x)e^{tc_x(\eta_x)}] \\ &\leq \frac{c_2 r}{c_1 r_x}\nu_{\Lambda,r}[e^{tc_x(\eta_x)}]\end{aligned}$$

Putting all of this together, and recalling that $|t| \leq 1$, we have

$$th'(t) - h(t)\log h(t) \leq C'' r t^2 h(t),$$

for some constant C'', which is the same as

$$\frac{d}{dt}\frac{\log h(t)}{t} \leq C'' r.$$

Integrating in t and noting that $\lim_{t\to 0}\frac{\log h(t)}{t} = \nu_{\Lambda,r}[c_x(\eta_x)]$ we obtain

$$\frac{\log h(t)}{t} \leq \nu_{\Lambda,r}[c_x(\eta_x)] + C'' r t,$$

from which the first part of the Proposition follows. The constant C'' depends on N via $M = BN$ of Lemma 4.3.

We now turn to the proof of

$$\nu_{\Lambda,r}\left[e^{t\cdot r\cdot\{h_x(\eta_x)-\nu_{\Lambda,r}[h_x(\eta_x)]\}}\right] \leq C(N)e^{C(N)\{rt^2+\sqrt{r}|t|\}}.$$

By the change of measure formula (6.9) we may calculate the expectation of $h_x(\eta_x) = \frac{\eta_x+1}{c_x(\eta_x+1)}$, to be

$$\nu_{\Lambda,r}[h_x(\eta_x)] = \frac{\nu_{\Lambda,r+1}[\eta_x]}{\nu_{\Lambda,r+1}[c_x(\eta_x)]} = \frac{\tilde{r}_x}{\nu_{\Lambda,r+1}[c_x(\eta_x)]}.$$

Hence,

$$\begin{aligned}|h_x(\eta_x) - \nu_{\Lambda,r}[h_x(\eta_x)]| &\leq \frac{1}{c_x(\eta_x+1)\nu_{\Lambda,r+1}[c_x(\eta_x)]}\Big\{c_x(\eta_x+1)\Big|\eta_x+1-\tilde{r}_x\Big| \\ &\qquad + (\eta_x+1)\Big|c_x(\eta_x+1) - \nu_{\Lambda,r+1}[c_x(\eta_x)]\Big|\Big\} \\ &\leq \frac{C'}{\tilde{r}_x}\left\{|c_x(\eta_x+1) - \nu_{\Lambda,r+1}[c_x(\eta_x)]| + |\eta_x+1-\tilde{r}_x|\right\},\end{aligned}$$

for some constant C' depending on c_1 and c_2. Using the uniform condtions (LG) and (M) we can show that for some $C > 0$ we have

$$|\eta_x + 1 - \tilde{r}_x| \leq C|c_x(\eta_x) - \nu_{\Lambda,r+1}[c_x(\eta_x)]| + C\sqrt{r},$$

which implies that

$$\begin{aligned}|h_x(\eta_x) - \nu_{\Lambda,r}[h_x(\eta_x)]| &\leq \frac{C}{\tilde{r}_x}\left\{C'|c_x(\eta_x+1) - \nu_{\Lambda,r+1}[c_x(\eta_x)]| + C\sqrt{\tilde{r}_x}\right\} \\ &\leq \frac{C'C}{r_x}\left\{|c_x(\eta_x) - \nu_{\Lambda,r}[c_x(\eta_x)]| + \sqrt{r}\right\}.\end{aligned}$$

We thus have that if $|c_x(\eta_x) - \nu_{\Lambda,r}[c_x(\eta_x)]| \leq M$ then $r|h_x(\eta_x) - \nu_{\Lambda,r}[h_x(\eta_x)]| \leq \frac{Cr}{r_x}(M+\sqrt{r}) \leq C(M+\sqrt{r})$. Therefore, for t in $(0,1]$

$$\begin{aligned}\nu_{\Lambda,r}[e^{tr(h_x(\eta_x)-\nu_{\Lambda,r}[h_x(\eta_x)])}] &= t\int e^{tz}\nu_{\Lambda,r}[r_x(h_x(\eta_x) - \nu_{\Lambda,r}[h_x(\eta_x)]) > z]dz \\ &\leq t\int e^{tz}\nu_{\Lambda,r}[c_x(\eta_x) - \nu_{\Lambda,r}[c_x(\eta_x)] > \frac{z}{C} - C\sqrt{r}]dz \\ &\quad +t\int e^{tz}\nu_{\Lambda,r}[c_x(\eta_x) - \nu_{\Lambda,r}[c_x(\eta_x)] < -\frac{z}{C} + C\sqrt{r}]dz\end{aligned} \tag{6.11}$$

By change of variable (6.11) is equal to

$$\begin{aligned}&Ct\int e^{t(CM+C\sqrt{r})}\nu_{\Lambda,r}[c_x(\eta_x) - \nu_{\Lambda,r}[c_x(\eta_x)] > M]dM \\ &+ Ct\int e^{t(CM+C\sqrt{r})}\nu_{\Lambda,r}[c_x(\eta_x) - \nu_{\Lambda,r}[c_x(\eta_x)] < -M]dM \\ &= Ce^{Ct\sqrt{r}}\left\{\nu_{\Lambda,r}[e^{Ct(c_x(\eta_x)-\nu_{\Lambda,r}[c_x(\eta_x)])}]\right. \\ &\qquad\qquad \left.+\nu_{\Lambda,r}[e^{-Ct(c_x(\eta_x)-\nu_{\Lambda,r}[c_x(\eta_x)])}]\right\} \\ &\leq Ce^{Ct\sqrt{r}}e^{Crt^2}.\end{aligned}$$

Replacing $h_x(\eta_x) - \nu_{\Lambda,r}[h_x(\eta_x)]$ with its negative gives us the same bound for t in $[-1,0)$. □

Because the constant $C(N')$ is allowed to depend on N' in any way, we may extend these calculations of Proposition 6.4 using Cauchy-Schwarz to say

$$\log\ \nu_{\Lambda',r'}\left(e^{t\cdot\sum_{x\in\Lambda'}(c_x(\eta_x)-\nu_{\Lambda,r}(c_x(\eta_x)))}\right) \leq C(N')\ r'\ t^2 \tag{6.12}$$

$$\nu_{\Lambda',r'}\left(e^{t\cdot r\cdot\sum_{x\in\Lambda'}(h_x(\eta_x)-\nu_{\Lambda,r}(h_x(\eta_x)))}\right) \leq C(N')e^{C(N')(r't^2+\sqrt{r'}|t|)}. \tag{6.13}$$

Proposition 6.5. *There exists a constant $C(N')$ (depending on $N' = |\Lambda'|$) such that*

$$\nu_{\Lambda',r'}\left[f;\sum_{x\in\Lambda'}c_x(\eta_x)\right]^2 \leq C(N')\cdot r'\cdot\nu_{\Lambda',r'}[f]H(\tilde{f}|\nu_{\Lambda',r'})$$

$$\nu_{\Lambda',r'}\left[f;\sum_{x\in\Lambda'}h_x(\eta_x)\right]^2 \leq \frac{C(N')}{r'}\nu_{\Lambda',r'}[f]\left\{\nu_{\Lambda',r'}[f] + H(\tilde{f}|\nu_{\Lambda',r'})\right\}$$

Sketch of proof. The proposition follows from direct calculation if in each case we insert the bounds (6.12) and (6.13) into the entropy inequality (6.8) and optimize over s. $\square$

From the above bounds we have the following

$$\nu_{\Lambda_1,r_1}\left[\nu_{\Lambda_2,r-r_1}[f];\sum_{x\in\Lambda_1}c_x(\eta_x)\right]^2 \le C(N)r_1\nu_{\Lambda,r}[f|R_1=r_1]H(\nu_{\Lambda_2,r-r_1}[f]|\nu_{\Lambda_1,r_1}) \tag{6.14}$$

as well as

$$\begin{aligned}&\nu_{\Lambda_2,r-r_1}\left[\nu_{\Lambda_1,r_1}[f];\sum_{y\in\Lambda_2}h_y(\eta_y)\right]^2\\ &\le \frac{C(N)}{r-r_1}\nu_{\Lambda,r}[f|R_1=r_1]\left\{\nu_{\Lambda,r}[f|R_1=r_1]+H(\nu_{\Lambda_1,r_1}[f]|\nu_{\Lambda_2,r-r_1})\right\}\end{aligned} \tag{6.15}$$

We now insert (6.14) and (6.15) into (6.7) to obtain (6.6). Notice that we have also used that the entropy is convex.

Step 4: putting it all together. We next use Proposition 5.9 which by symmetry also applies to γ_2, from which it follows that

$$\frac{\gamma_2^2(r-r_1)}{\gamma_2^2(r-r_1+1)}\frac{r_1^2}{(r_2+1)^3}\le\frac{C}{r_2+1}.$$

We insert this into (6.6) to obtain

$$\begin{aligned}&Cr\ \frac{\gamma_1(r_1)\wedge\gamma_1(r_1-1)}{\nu_{\Lambda,r}(f|R_1=r_1)\vee\nu_{\Lambda,r}(f|R_1=r_1-1)}\ B^2(r_1)\\ &=Cr\ \frac{\gamma_2(r_2)\wedge\gamma_2(r_2+1)}{\nu_{\Lambda,r}(f|R_1=r_1)\vee\nu_{\Lambda,r}(f|R_1=r_1-1)}\ B^2(r_1)\\ &\le C(N)\gamma_1(r_1)\left(\nu_{\Lambda,r}(f|R_1=r_1)\right.\\ &\qquad\left.+\nu_{\Lambda_2,r-r_1}(H(f|\nu_{\Lambda_1,r_1}))+\nu_{\Lambda_1,r_1}(H(f|\nu_{\Lambda_2,r-r_1}))\right)\end{aligned} \tag{6.16}$$

where we have also used that $r_1\le\frac{r}{2}$. We obtain a similar answer in the case $r_1>\frac{r}{2}$.

This gives us the necessary bounds on the term B. We combine this with Proposition 6.2 which gives us bounds on the term A, and insert into (6.1):

$$\begin{aligned}
&H(\nu_{\Lambda,R}(f|R_1)|\nu_{\Lambda,r})\\
&\le Cr\sum_{r_1=1}^{r}\frac{\gamma(r_1)\wedge\gamma(r_1-1)}{\nu_{\Lambda,r}(f|R_1=r_1)\vee\nu_{\Lambda,r}(f|R_1=r_1-1)}[A^2(r_1)+B^2(r_1)]\\
&\le C(N)\sum_{r_1=1}^{r}\gamma(r_1-1)\Big\{D_{\Lambda,r}(\sqrt{f})+\nu_{\Lambda,r}(f|r_1)\\
&\qquad\qquad +\nu_{\Lambda_2,r-r_1}[H(f|\nu_{\Lambda_1,r_1})]+\nu_{\Lambda_1,r_1}[H(f|\nu_{\Lambda_2,r-r_1})]\Big\}
\end{aligned}$$

for some possibly different constant $C(N)$ depending again only on N. We next apply the induction hypothesis to obtain the new bound

$$C(N)[D_{\Lambda,r}(\sqrt{f})+\nu_{\Lambda,r}(f)+\kappa(N,r)D_{\Lambda,r}(\sqrt{f})]$$

where $\kappa(N,r)$ was defined in (3.2). This completes the argument required to prove (3.4).

Proposition 6.6.

$$\sup_r \kappa(2,r)<\infty.$$

Proof. We assume that $\Lambda=\{0,1\}$. In this case, since there is a total of r particles, the function $f(\eta)=f(k,r-k)=\tilde{f}(k)$. We begin by calculating the Dirichlet form

$$\begin{aligned}
D_{2,r}\left(\sqrt{\tilde{f}}\right) &= \sum_{k=0}^{r}\gamma_1(k)c_1(k)\left[\sqrt{\tilde{f}(k-1)}-\sqrt{\tilde{f}(k)}\right]\\
&\quad+\sum_{k=0}^{r}\gamma_1(k)c_2(r-k)\left[\sqrt{\tilde{f}(k+1)}-\sqrt{\tilde{f}(k)}\right]\\
&= \sum_{k=1}^{r}\gamma_1(k)c_1(k)\left[\sqrt{\tilde{f}(k-1)}-\sqrt{\tilde{f}(k)}\right]
\end{aligned}$$

using the relationship $\gamma_1(k)c_2(r-k)=\gamma_1(k+1)c_1(k+1)$. We next prove that there exists a finite constant B so that $r\ \gamma_1(k)\wedge\gamma_1(k-1)\le B\gamma_1(k)c_1(k)$.

$$\frac{\gamma_1(k)c_1(k)}{\gamma_1(k)\wedge\gamma_1(k-1)}=c_1(k)\vee c_2(r-k-1)\ge\frac{1}{B}\frac{r}{2}.$$

We may now put these results together to obtain

$$\begin{aligned} H(f|\nu_{2,r}) &= H(\tilde{f}|\gamma_1(\cdot)) \\ &\leq Cr\sum_{k=1}^{r}\gamma_1(k)\vee\gamma_1(k-1)\left[\sqrt{\tilde{f}(k-1)}-\sqrt{\tilde{f}(k)}\right] \\ &\leq CB\sum_{k=1}^{r}\gamma_1(k)c_1(k)\left[\sqrt{\tilde{f}(k-1)}-\sqrt{\tilde{f}(k)}\right] \\ &= CBD_{2,r}(\sqrt{f}). \end{aligned}$$

We also used Proposition 5.4 in the above. This completes the proof. □

7. Tightening the Bounds

As discussed earlier, in this section we obtain improved bounds on the covariances appearing in Proposition 6.5, which will allow us to conclude that (3.8) holds for large values of N:

$$H(\nu_{\Lambda,R}(f|R_1=r_1)|\nu_{\Lambda,r}) \leq CN^2D_{\Lambda,r}(\sqrt{f})+C\nu_{\Lambda,r}[f]+\kappa(N,r)D_{\Lambda,r}(\sqrt{f}).$$

The two tighter bounds on the covariances in Proposition 6.5 are given below.

Proposition 7.1. *For every $\epsilon>0$ there exists a constant $C=C(\epsilon)>0$ and an $N_0=N_0(\epsilon)$ such that for all $|\Lambda'|=N\geq N_0$, all r' and all positive functions f*

$$\nu_{\Lambda',r'}[f;\sum_{x\in\Lambda'}c_x(\eta_x)]^2 \leq r'\nu_{\Lambda',r'}[f]\left[C\nu_{\Lambda',r'}[f]+CN^2D_{\Lambda',r}(\sqrt{f})+\epsilon H(f|\nu_{\Lambda',r})\right].$$

Proposition 7.2. *For every $\epsilon>0$ there exists a constant $C=C(\epsilon)>0$ such that for all $|\Lambda'|=N\geq N_0$, all r' and all positive functions f*

$$\nu_{\Lambda',r'}[f;\sum_{x\in\Lambda'}h_x(\eta_x)]^2 \leq \frac{N^2}{r'}\nu_{\Lambda',r'}[f]\left[C\nu_{\Lambda',r'}[f]+CN^2D_{\Lambda',r'}(\sqrt{f})+\epsilon H(f|\nu_{\Lambda',r'})\right].$$

Let us first show how these bounds give the desired result. We may assume that both N_0 are the same in the propositions. Using the same argument as in Step 4 of the previous section, but replacing the covariance bounds of Proposition 6.5 with the bounds of the above Propositions, we estimate $B^2(r_1)$. The new estimates give

$$\begin{aligned} &r\,\frac{\gamma_1(r_1)\wedge\gamma_1(r_1-1)}{\nu_{\Lambda,r}(f|R_1=r_1)\vee\nu_{\Lambda,r}(f|R_1=r_1-1)}\,B^2(r_1) \\ &\leq \gamma_1(r_1)\left(C(\epsilon)\nu_{\Lambda,r}(f|R_1=r_1)+C(\epsilon)N^2D_{\nu_{\Lambda_2,r-r_1}}(\sqrt{f})+\epsilon H(f|\nu_{\Lambda_2,r-r_1})\right). \end{aligned}$$

We now combine the above result along with Proposition 6.2 to continue with line (6.1):

$$
\begin{aligned}
&H(\nu_{\Lambda,R}(f|R_1)|\nu_{\Lambda,r}) \\
&\leq Cr\sum_{r_1=1}^{r} \frac{\gamma_1(r_1)\wedge\gamma_1(r_1-1)}{\nu_{\Lambda,r}(f|R_1=r_1)\vee\nu_{\Lambda,r}(f|R_1=r_1-1)}[A^2(r_1)+B^2(r_1)] \\
&\leq C\sum_{r_1=1}^{r}\gamma_1(r_1)\Big\{N^2 D_{\Lambda,r}(\sqrt{f})+C(\epsilon)\nu_{\Lambda,r}(f|r_1) \\
&\qquad\qquad +C(\epsilon)N^2 D_{\nu_{\Lambda_2,r-r_1}}(\sqrt{f})+\epsilon H(f|\nu_{\Lambda_2,r-r_1})\Big\} \\
&\leq C(\epsilon)N^2 D_{\Lambda,r}(\sqrt{f})+C(\epsilon)\nu_{\Lambda,r}(f)+C\cdot\epsilon H(f|\nu_{\Lambda_2,r-r_1})
\end{aligned}
$$

We now fix an ϵ so that $C\cdot\epsilon<1$. The above together with (3.2) gives the bound

$$
CN^2 D_{\Lambda,r}(\sqrt{f})+C\nu_{\Lambda,r}(f)+\kappa(N,r)D_{\Lambda,r}(\sqrt{f}),
$$

for some new constant C, and $|\Lambda|\geq 2N_0$. This is (3.8) as required.

The rest of this section is divided as follows. We describe in detail the proof of Proposition 7.1, which is split into two main cases: small and large density. We then proceed with the proof of Proposition 7.2, which follows by a similar argument. For ease of presentation, we will write Λ for Λ' and r for r' in both proofs.

7.1. **Proof of Proposition 7.1.** The proof of this result is split into several lemmas. We begin by partitioning Λ into m disjoint blocks $\Lambda_1,\ldots,\Lambda_m$, which we assume, without loss of generality, to be of equal size $l=N/m$. Denote by $\mathcal{G}$ the σ-field generated by $R_1,\ldots,R_m$, where R_i is the random number of particles inside the subset Λ_i. We thus obtain

$$
\begin{aligned}
&\nu_{\Lambda,r}[f;\sum_{x\in\Lambda}c_x(\eta_x)] \\
&=\nu_{\Lambda,r}[\nu_{\Lambda,r}[f;\sum_{x\in\Lambda}c_x(\eta_x)|\mathcal{G}]]+\nu_{\Lambda,r}[f;\sum_{k=1}^{m}\nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k}c_x(\eta_x)]] \qquad (7.1)
\end{aligned}
$$

and we bound the left hand side and the right hand side of (7.1) separately. The bound on the left hand side is easier and its proof is essentially a restatement of the proof of the first part of Proposition 6.5 .

Proposition 7.3. *There is a constant C, possibly depending on l, such that*

$$
\nu_{\Lambda,r}[\nu_{\Lambda,r}[f;\sum_{x\in\Lambda}c_x(\eta_x)|\mathcal{G}]]^2\leq Cr\nu_{\Lambda,r}[f]\nu_{\Lambda,r}[H(f|\nu_{\Lambda,r}(\cdot|\mathcal{G}))].
$$

Proof. We begin with the entropy inequality; for any $t > 0$ we have

$$\begin{aligned}\nu_{\Lambda,r}[f;\sum_{x\in\Lambda} c_x(\eta_x)|\mathcal{G}] \quad &\leq \quad \frac{\nu_{\Lambda,r}[f]}{t}\sum_{k=1}^{m}\log\nu_{\Lambda_k,R_k}[\exp\{t\sum_{x\in\Lambda_k}(c_x(\eta_x)-\nu_{\Lambda_k,R_k}[c_x(\eta_x)])\}] \\ &\qquad +\frac{1}{t}H(f|\nu_{\Lambda,r}[\cdot|\mathcal{G}]).\end{aligned}$$

Using the Cauchy-Schwarz inequality and Proposition 6.4 we have the following bound

$$\nu_{\Lambda_k,R_k}[\exp\{t\sum_{x\in\Lambda_k}(c_x(\eta_x)-\nu_{\Lambda_k,R_k}[c_x(\eta_x)])\}] \leq \exp\{c(l)R_k t^2\},$$

for some constant $c(l)$ depending on l. Combining the two inequalities we then have for any $t > 0$

$$\nu_{\Lambda,r}[\nu_{\Lambda,r}[f;\sum_{x\in\Lambda} c_x(\eta_x)|\mathcal{G}]]^2 \leq \nu_{\Lambda,r}[f]^2 c(l) r^2 t^2 + \frac{1}{t^2}\left(\nu_{\Lambda,r}[H(f|\nu_{\Lambda,r}(\cdot|\mathcal{G}))]\right)^2$$

The result follows if we optimize in t. $\square$

The bounds on the right hand side of (7.1) are considerably more difficult. These are given in the following lemma.

Proposition 7.4. *For every $\epsilon > 0$ there is an $l = l(\epsilon)$, $N_0 = N_0(\epsilon)$, and a constant $C = C(\epsilon) > 0$ such that for all $N \geq N_0$*

$$\nu_{\Lambda,r}[f;\sum_{k=1}^{m}\nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)]]^2 \leq r\nu_{\Lambda,r}[f]\left\{C\nu_{\Lambda,r}[f] + CN^2 D_{\Lambda,r}(\sqrt{f}) + \epsilon H(f|\nu_{|\Lambda,r})\right\}.$$

Notice that from Section 6 we know that logarithmic Sobolev constant κ depends only the the size of the subset (and not on the number of particles). We apply this to obtain the bound

$$\begin{aligned}\nu_{\Lambda,r}[H(f|\nu_{\Lambda,r}(\cdot|\mathcal{G}))] \quad &= \quad \nu_{\Lambda,r}[\sum_{k=1}^{m} H(f|\nu_{C_k,R_k}(\cdot))] \\ &\leq \quad \nu_{\Lambda,r}[\sum_{k=1}^{m}\kappa(l) D_{C_k,R_k}(\sqrt{f})] \\ &\leq \quad C D_{\Lambda,r}(\sqrt{f})\end{aligned}$$

for a constant C depending on l. From this it follows that Propositions 7.3 and 7.4 together imply Proposition 7.1. We next prove Proposition 7.4. We split it up into several cases, depending on the size of ρ. Up to now our estimates have relied largely on either one-site bounds or bounds using the local central limit theorem. Because of this the proofs have been similar to the non-homogeneous case. However, because of the two-blocks estimates, the proofs now rely on the joint behaviour over the boxes.

The methods developed in [DPP2] still apply, however, with slight modifications. We begin with some initial estimates.

Lemma 7.5. (i) *For every $\varphi > 0$ and $t \in \mathbb{R}$*

$$\mu_{\Lambda,\varphi}[e^{t(c_x(\eta_x)-\varphi)}] \le e^{\varphi a_1 t^2 e^{a_1|t|}}$$

(ii) *There exists a $C > 0$ so that*

$$\mu_{\Lambda,\varphi}[e^{t\eta_x}] \le e^{Ct\rho e^{Ct}}.$$

Proof. In the first inequality we repeat the argument of (6.4). Let $h(t) = \mu_{\Lambda,\varphi}[e^{tc_x(\eta_x)}]$. Also we remind the reader of the inequality due to assumption (LG) $|c_x(k+1) - c_x(k)| \le a_1$. By a simple change of measure

$$\mu_{\Lambda,\varphi}[c_x(\eta_x)f(\eta_x)] = \varphi\mu_{\Lambda,\varphi}[f(\eta_x+1)]$$

and Jensen's inequality we obtain

$$\begin{aligned} th'(t) - h(t)\log h(t) &= t\mu_{\Lambda,\varphi}[c_x e^{tc_x}] - \mu_{\Lambda,\varphi}[e^{tc_x}]\log\mu_{\Lambda,\varphi}[e^{tc_x}] \\ &\le \varphi t\mu_\varphi[e^{tc_x(\eta_x+1)} - e^{tc_x(\eta_x)}] \\ &\le a_1\varphi t^2 e^{a_1 t}\mu_\varphi[e^{tc_x}]. \end{aligned}$$

We used the inequality $|e^x - e^y| \le |x-y|e^{|x-y|}e^{|y|}$. Because

$$th'(t) - h(t)\log h(t) = t^2 h(t)\partial_t \frac{\log h(t)}{t}$$

this translates to

$$\partial_t \frac{\log h(t)}{t} \le \varphi a_1 e^{a_1 t},$$

where $\lim_{t\to 0}\frac{\log h(t)}{t} = \varphi$. Integrating we thus have that

$$h(t) = \mu_\varphi[e^{tc_x}] \le e^{\varphi t e^{a_1 t}},$$

which implies

$$\mu_\varphi[e^{t(c_x-\varphi)}] \le e^{\varphi t(e^{a_1 t}-1)} \le e^{\varphi a_1 t^2 e^{a_1 t}}.$$

The bounds on $h(t)$ along with the fact that $c_2 \ge \frac{c_x(k)}{k} \ge c_1$ imply the second inequality. $\square$

We also need a result similar to (i) above for η_x.

Lemma 7.6. *There exists a $C > 0$ so that*

$$\mu_{\Lambda,\varphi}\left[e^{t|\eta_x-\rho_x|}\right] \le Ce^{C(t\sqrt{\rho_x}+\rho_x t^2 e^{C|t|})}$$

for all $\varphi > 0$.

Proof. We first use conditions (LG) and (M) to obtain

$$\begin{aligned}|\eta_x - \rho_x| &\leq C\{|c_x(\eta_x) - c(\rho_x)| + 1\}\\ &\leq C\{|c_x(\eta_x) - \varphi(\rho_x)| + |c_x(\rho_x) - \varphi(\rho_x)| + 1\}\\ &\leq C\{|c_x(\eta_x) - \varphi(\rho_x)| + \sqrt{\rho_x} + 1\}\end{aligned}$$

where the last inequality

$$|c_x(\rho_x) - \varphi(\rho_x)| \leq \sqrt{\rho_x}$$

is proved as in [LSV]:

$$|\varphi(\rho_x) - c_x(\rho_x)| \leq \mu_{\varphi_x}[|c_x(\eta_x) - c_x(\rho_x)|] \leq a_1\mu_{\varphi_x}[|\eta_x - \rho_x|] \leq a_1\sigma_x(\rho_x).$$

The remainder now follows from Proposition 4.4 and Lemma 7.5. □

We continue with the proof of Proposition 7.4. As mentioned previously, we split this into two cases: large density $\rho > \rho_0$ and small density $\rho \leq \rho_0$.

7.1.1. *Case 1. large density:* $\rho > \rho_0$. For ease of calculation, and without loss of generality, we may assume that $\nu_{\Lambda,r}[f] = 1$. We begin with the entropy inequality:

$$\begin{aligned}&\nu_{\Lambda,r}[f; \sum_{k=1}^{m} \nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)]]\\ &\leq \frac{1}{t}\log \nu_{\Lambda,r}[e^{t\sum_{k=1}^{m}\left\{\nu_{R_k,\Lambda_k}\left[\sum_{x\in\Lambda_k} c_x(\eta_x)\right] - \nu_{r,\Lambda}\left[\sum_{x\in\Lambda_k} c_x(\eta_x)\right]\right\}}] - \frac{1}{t}H(f|\nu_{\Lambda,r}) \qquad (7.2)\end{aligned}$$

The next steps will focus on bounding the expectation inside the logarithm. Here is where the first difference from the proof of the homogeneous case appears. The tighter bounds are achieved by applying a Taylor series type argument to the function φ on the boxes Λ_k.

In what follows, unless otherwise specified, let $\varphi = \varphi_\Lambda(\frac{r}{|\Lambda|})$ and $\rho = \frac{r}{|\Lambda|}$. We let $r_k = \nu_{\Lambda,r}[R_k]$, where R_k is the number of particles in Λ_k and we also define $\rho_k = \mu_{\Lambda,\varphi}[AV_{x\in\Lambda_k}\eta_x]$. We will also denote φ_{Λ_k} as φ_k. Notice that $\varphi_k(\rho_k) = \varphi(\rho)$.

We define the function

$$\tilde{c}_x(m) = c_x(m) - \varphi_k'(\rho_k)\cdot m, \qquad (7.3)$$

for all $x \in \Lambda_k$. For the time being it is enough to know that $\varphi_k'(x)$ is a strictly positive quantity uniformly bounded in x for all k. We next bound

$$\nu_{\Lambda,r}[e^{t\sum_{k=1}^{m}\left\{\nu_{R_k,\Lambda_k}\left[\sum_{x\in\Lambda_k} c_x(\eta_x)\right] - \nu_{r,\Lambda}\left[\sum_{x\in\Lambda_k} c_x(\eta_x)\right]\right\}}] \leq H_1 \times H_2$$

where

$$\begin{aligned}H_1 &= e^{-t\sum_{k=1}^{m}\left\{\left[\nu_{r,\Lambda}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right] - \mu_\varphi\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]\right] + \varphi_k'(\rho_k)[r_k - l\rho_k]\right\}}\\ H_2 &= \nu_{\Lambda,r}[e^{t\sum_{k=1}^{m}\left\{\left(\nu_{R_k,\Lambda_k}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right] - \mu_\varphi\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]\right) + \varphi_k'(\rho_k)(R_k - \rho_k)\right\}}]. \qquad (7.4)\end{aligned}$$

From Propositions 4.4 and 4.10, and from the Cauchy-Schwarz inequality we have that

$$\begin{aligned} |\nu_{\Lambda,r}[c_x(\eta_x)] - \varphi| &\leq C\frac{1}{|\Lambda|}\sqrt{\rho_x}, \quad \text{and} \\ |r_k - l\rho_k| &\leq C\frac{l}{|\Lambda|}\sqrt{l\rho_k}, \end{aligned}$$

for $|\Lambda|$ sufficiently large, from which it follows that

$$H_1 \leq e^{Ct\sqrt{r}}. \tag{7.5}$$

Next, by the Cauchy-Schwarz inequality

$$\begin{aligned} H_2 \;\leq\; & \nu_{\Lambda,r}\big[e^{2t\sum_{k=1}^m \left\{\nu_{R_k,\Lambda_k}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]-\mu_\varphi\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]\right\}}\big]^{1/2} \\ & \times\;\nu_{\Lambda,r}\big[e^{2t\sum_{k=1}^m\left\{\varphi_k'(\rho_k)(R_k-\rho_k)\right\}}\big]^{1/2}. \end{aligned} \tag{7.6}$$

The second term of these satisfies the following inequality by applying Cauchy-Schwarz again

$$\begin{aligned} \nu_{\Lambda,r}\big[e^{t\sum_{k=1}^m\left\{\varphi_k'(\rho_k)(R_k-l\rho_k)\right\}}\big] \;\leq\; & \nu_{\Lambda,r}\big[e^{2t\sum_{k\leq m/2}\left\{\varphi_k'(\rho_k)(R_k-l\rho_k)\right\}}\big]^{1/2} \\ & \times\nu_{\Lambda,r}\big[e^{2t\sum_{k>m/2}\left\{\varphi_k'(\rho_k)(R_k-l\rho_k)\right\}}\big]^{1/2}. \end{aligned}$$

We next apply Proposition 4.9 to obtain that this is bounded above by

$$C\left\{\Pi_{k=1}^m\Pi_{x\in\Lambda_k}\mu_\varphi\big[e^{2t\varphi_k'(\rho_k)(\eta_x-\rho_x)}\big]\right\}^{\frac{1}{2}}.$$

We bound this last quantity using the estimates of Lemma 7.6 by

$$\nu_{\Lambda,r}\big[e^{t\sum_{k=1}^m\left\{\varphi_k'(\rho_k)(R_k-l\rho_k)\right\}}\big] \leq e^{Ct\sqrt{r}+c\rho t^2e^{Ct}}. \tag{7.7}$$

Therefore it remains to bound the first part of line (7.6). This is where the Taylor argument becomes important. Notice that because

$$\rho_k = \mu_{\varphi(\rho)}[AV_{x\in\Lambda_k}\eta_x],$$

we have that $\varphi_k(\rho_k) = \varphi(\rho)$. We have set up a two-block argument, and we would like to work in measures μ on Λ_k where the underlying density is $r_k = \frac{R_k}{l}$. With this in mind, and in a slight abuse of notation, let φ_k denote $\varphi_k\left(\frac{R_k}{l}\right) = \varphi_{\Lambda_k}\left(\frac{R_k}{l}\right)$. By the same argument as above, we may use the Cauchy-Schwarz inequality together with Proposition 4.9 to obtain

$$\begin{aligned} & \nu_{\Lambda,r}\left[e^{t\sum_{k=1}^m\left\{\left(\nu_{R_k,\Lambda_k}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]-\mu_\varphi\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]\right)\right\}}\right] \\ & \quad\leq\; C\mu_\varphi\left[e^{2t\sum_{k=1}^m\left\{\nu_{R_k,\Lambda_k}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]-\mu_\varphi\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]\right\}}\right]^{1/2} \\ & \quad\leq\; C\mu_\varphi\left[e^{4t\sum_{k=1}^m\bar{Y}_k}\right]^{1/4}\mu_\varphi\left[e^{4t\sum_{k=1}^m\bar{W}_k}\right]^{1/4}, \end{aligned}$$

where we set

$$\begin{aligned} Y_k &= \nu_{R_k,\Lambda_k}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right] - \mu_{\varphi_k}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right] \qquad \text{and} \\ W_k &= \mu_{\varphi_k}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right] - \mu_{\varphi}\left[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)\right]. \end{aligned}$$

We let $\bar{Y}_k$ and $\bar{W}_k$ denote the centered versions of Y_k and W_k under the measure μ_φ. We first bound $\mu_\varphi[e^{t\sum_{k=1}^m \bar{Y}_k}]$. Notice the following inequality

$$e^x = \sum_{k\geq 0}\frac{x^k}{k!} \leq 1 + x + \frac{x^2}{2}\sum_{k\geq 0}\frac{|x|^k}{k!} \leq 1 + x + \frac{x^2}{2}e^{|x|}.$$

This implies that $\exp\{x\} \leq \exp\{x + \frac{x^2}{x}e^{|x|}\}$. Applying this inequality to $E[e^X] \leq e^{\{E[X]+E[X^2e^{|X|}]\}}$ we have

$$\mu_\varphi\left[e^{4t\bar{Y}_k}\right] \leq \exp\left\{Ct^2\mu_\varphi\left[\bar{Y}_k^2 e^{t|\bar{Y}_k|}\right]\right\}.$$

We next bound Y_k by $C\sqrt{1+\frac{R_k}{l}}$, for some positive constant C depending on l. This is a consequence of Proposition 4.11, and holds for l sufficiently large. The presence of the extra 1 comes from not being able to bound R_k from below. We plug this into the above to obtain

$$\begin{aligned} &\exp\left\{Ct^2\mu_\varphi\left[\left(1+\frac{R_k}{l}\right)e^{t\sqrt{1+\frac{R_k}{l}}}\right]\right\} \\ &\leq \exp\left\{Ct^2\mu_\varphi\left[\left(1+\frac{R_k}{l})^2\right]^{1/2}\mu_\varphi[e^{2t\sqrt{1+\frac{R_k}{l}}}\right]^{1/2}\right\} \\ &\leq \exp\left\{C(\rho_0)t^2\rho_k\mu_\varphi\left[e^{2t\sqrt{1+\frac{R_k}{l}}}\right]^{1/2}\right\}. \end{aligned} \tag{7.8}$$

We next bound $\mu_\varphi\left[e^{2t\sqrt{1+\frac{R_k}{l}}}\right]^{1/2}$. We make use of the following lemma proved in [DPP2].

Lemma 7.7. *Suppose that for a random variable $X \geq 0$ and a function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ we have that for all $t \geq 0$ we*

$$E[e^{tX}] \leq e^{tg(t)E[X]}. \tag{7.9}$$

Then, for all $t \geq 0$

$$E[e^{t\sqrt{X}}] \leq \exp\{t\sqrt{g(2t)+g(t)}\sqrt{E[X]}\} + e^t.$$

Proof. Using the inequality $\sqrt{x} \leq x+1$ as well as Cauchy-Schwarz and Chebychev's inequalities we have

$$\begin{aligned} E[e^{t\sqrt{X}}] &\leq E[e^{t\sqrt{X}}\mathbb{I}[X < kE[X]]] + E[e^{t\sqrt{X}}\mathbb{I}[X \geq kE[X]]] \\ &\leq e^{t\sqrt{kE[X]}} + e^t E[e^{tX}\mathbb{I}[X \geq kE[X]]] \\ &\leq e^{t\sqrt{kE[X]}} + e^t E[e^{2tX}]^{1/2} P[X \geq kE[X]]^{1/2} \\ &\leq e^{t\sqrt{kE[X]}} + e^t E[e^{2tX}]^{1/2} \left[\frac{E[e^{tX}]}{e^{tkE[X]}}\right]^{1/2}. \end{aligned}$$

Applying twice the assumption (7.9) we have that

$$\begin{aligned} e^t E[e^{2tX}]^{1/2} [\frac{E[e^{tX}]}{e^{tkE[X]}}]^{1/2} &\leq e^{\{t+\frac{1}{2}g(2t)E[X]\}} \left[\frac{e^{tg(t)E[X]}}{e^{tkE[X]}}\right]^{1/2} \\ &= e^t e^{\frac{t}{2}E[X]\{g(2t)+g(t)-k\}}. \end{aligned}$$

The desired result follows if we choose $k = g(2t) + g(t)$. □

The second part of Lemma 7.5 gives that

$$\mu_\varphi[e^{2t\frac{R_k}{l}}] \leq e^{C\rho_k t e^{Ct}}.$$

We may now use Lemma 7.7 with $g(t) = e^{Ct}$.

$$\begin{aligned} \mu_\varphi[e^{2t\sqrt{1+\frac{R_k}{l}}}] &\leq e^{2Ct}\mu_\varphi[e^{2Ct\sqrt{\frac{R_k}{l}}}] \\ &\leq e^{2Ct}(e^{Ce^t\sqrt{\rho_k}t} + e^{Ct}). \end{aligned}$$

We insert this result into (7.8) and sum over k to get the following bound

$$\mu_\varphi[e^{t\sum_{k=1}^m \bar{Y}_k}] \leq e^{C\rho t^2 e^{Ct\sqrt{\rho}e^{Ct}}}. \tag{7.10}$$

We now consider the remaining term $\mu_\varphi[e^{t\sum_{k=1}^m \bar{W}_k}]$. We first write $W_k = lF_k$ where

$$\begin{aligned} F_k &= \varphi_k\left(\frac{R_k}{l}\right) - \varphi(\rho) - \varphi_k'(\rho_k)\left(\frac{R_k}{l} - \rho_k\right) \\ &= \varphi_k\left(\frac{R_k}{l}\right) - \varphi_k(\rho_k) - \varphi_k'(\rho_k)\left(\frac{R_k}{l} - \rho_k\right) \end{aligned} \tag{7.11}$$

and define

$$Z_k = \frac{1}{l}\sum_{x\in\Lambda_k} \frac{\eta_x - \rho_x}{\sigma_k(\varphi)}. \tag{7.12}$$

Notice that $\sigma_k Z_k = \frac{R_k}{l} - \rho_k$. Using the inequalities $e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}$ and $1 + x \leq e^x$ as before, as well as Cauchy-Schwarz, we get

$$\log \mu_\varphi[e^{tlF_k}] \leq \frac{1}{2}t^2 l^2 (\mu_{\Lambda,\varphi}[F_k^4])^{1/2} (\mu_{\Lambda,\varphi}[e^{2tl|F_k|}])^{1/2}. \tag{7.13}$$

Fix a positive constant B. If $|Z_k| > B$ we have that $|F_k| \leq C\sigma_k |Z_k|$, for some C, in which case

$$\begin{aligned}\mu_\varphi[F_k^4 \mathbb{I}\{|Z_k| > B\}] &\leq C\sigma^4 \mu_{\Lambda,\varphi}[Z_k^4 \mathbb{I}\{|Z_k| > B\}] \\ &\leq C\frac{\sigma^4}{B^4}\mu_{\Lambda,\varphi}[Z_k^8] \end{aligned} \tag{7.14}$$

where we write σ^2 in lieu of σ_Λ^2. If $|Z_k| \leq B$ we may bound $|F_k| \leq C(B)\sigma_k Z_k^2$ by using a Taylor argument, which also gives us

$$\mu_\varphi[F_k^4 \mathbb{I}\{|Z_k| \leq B\}] \leq C(B)\sigma^4 \mu_{\Lambda,\varphi}[Z_k^8]. \tag{7.15}$$

From Proposition 4.5 it follows that $\mu_{\Lambda,\varphi}[Z_k^8] \leq Cl^4$ for some constant C. Using these last results now along with Proposition 4.4 we obtain that the first part of (7.13) is bounded by

$$\mu_{\Lambda,\varphi}[F^4] \leq C(B)\frac{\rho^2}{l^4}.$$

Using the estimate $|F_k| \leq C\sigma_k |Z_k|$ we get

$$\begin{aligned}\mu_{\Lambda,\varphi}[e^{2tl|F_k|}] &\leq \mu_{\Lambda,\varphi}[e^{2t\sum_{x\in C_k}|\eta_x - \rho_x|}] \\ &\leq e^{Ctl\sqrt{\rho}} e^{Ct^2 l\rho e^{Ct}}\end{aligned}$$

using (5.8). These last two statements imply that (7.13) is bounded above by

$$\log \mu_\varphi[e^{tlF_k}] \leq C\rho t^2 e^{Ctl\sqrt{\rho}} e^{Ct^2 l\rho e^{Ct}}. \tag{7.16}$$

We now combine the bound on H_1 from (7.5) and the bounds on H_2 from (7.6), line (7.10), as well as (7.16) to obtain

$$\nu_{\Lambda,r}\big[e^{t\sum_{k=1}^n\{(\nu_{R_k,\Lambda_k}[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)] - \mu_\varphi[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)])\}}\big] \leq Ce^{Ct\sqrt{r}} e^{Ct^2\frac{r}{l}e^{Ctl\sqrt{\rho}}e^{Ct^2 l^2 \rho e^{Ctl}}} \tag{7.17}$$

where $\rho = \frac{r}{N}$. Combining (7.17) together with the entropy inequality (7.2) we obtain that for any $t > 0$

$$\nu_{\Lambda,r}[f; \sum_{k=1}^m \nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)]]^2 \leq \frac{C}{t^2} + Cr + ct^2\frac{r^2}{l^2}e^{Ctl\sqrt{\rho}}e^{Ct^2 l^2\rho e^{Ctl}} + \frac{1}{t^2}H^2(f|\nu_{\Lambda,r}) \tag{7.18}$$

We would now like to optimize the above inequality in t, as we have done before. However, due to the presence of the additional exponential terms (in comparison with the initial bounds on the covariances), this is considerably more difficult. We hence use a different approach. We choose t such that

$$t^2 = \frac{1 \vee [MH(f|\nu_{\Lambda,r})]}{r},$$

for any fixed M, and obtain the necessary bounds for three regimes on t.

Case 1(a): $t \leq \frac{M}{l\sqrt{\rho}} \wedge M$**.**

In this case we obtain the bound

$$\nu_{\Lambda,r}[f; \sum_{k=1}^{m} \nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)]]^2 \leq Cr + C\left\{\frac{M}{l} e^{CM+CM^2 e^{CM}} + \frac{1}{M}\right\} rH(f|\nu_{\Lambda,r}) \tag{7.19}$$

Case 1(b): $t > M$**.**

In this setting very rough approximations give

$$\nu_{\Lambda,r}[f; \sum_{x\in\Lambda} c_x(\eta_x)]^2 \leq Cr^2 \leq \frac{Cr}{M} H(f|\nu_{\Lambda,r}), \tag{7.20}$$

since $\sum_{x\in\Lambda} c_x(\eta_x) \leq Cr$. We used that $t > M$ implies that $rM \leq H$. This is clearly a tighter bound than the one in case 1 (a).

Case 1(c): $\frac{M}{l\sqrt{\rho}} \wedge M < t < M$**.**

Notice that this setting implies that

$$1 < l\sqrt{\frac{\rho}{Mr} H(f|\nu_{\Lambda,r})}. \tag{7.21}$$

This last case is the most complicated of the three, and requires its own approach. As before we have $\varphi_k = \varphi_k\left(\frac{R_k}{l}\right)$. In what follows we continue to assume that $\nu_{\Lambda,r}[f] = 1$. The function $\tilde{c}_x(k) = c_x(k) - \varphi_k'(\rho_k)k$ is the same function as before.

By carefully adding and subtracting terms, and noting that $\nu_{\Lambda,r}[f] = 1$, we obtain the following

$$\left|\nu_{\Lambda,r}\left[f; \sum_{k=1}^{m} \nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)]\right]\right|$$

$$\leq \left|\nu_{\Lambda,r}\left[f \cdot \sum_{k=1}^{m} \left\{\mu_{\varphi_k}[\sum_{x\in\Lambda_k} \tilde{c}_x(\eta_x)] - \mu_{\Lambda,\varphi}[\sum_{x\in\Lambda_k} \tilde{c}_x(\eta_x)]\right\}\right]\right| \tag{7.22}$$

$$+ \left|\nu_{\Lambda,r}\left[f \cdot \sum_{k=1}^{m} \varphi_k'(\rho_k)\left\{R_k - \rho_k \cdot l\right\}\right]\right| \tag{7.23}$$

$$+ \left|\nu_{\Lambda,r}\left[\sum_{k=1}^{m} \left\{\mu_{\varphi_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)] - \mu_{\Lambda,\varphi}[\sum_{x\in\Lambda_k} c_x(\eta_x)]\right\}\right]\right| \tag{7.24}$$

$$+ \left|\nu_{\Lambda,r}\left[f \cdot \sum_{k=1}^{m} \left\{\nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)] - \mu_{\varphi_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)]\right\}\right]\right| \tag{7.25}$$

The terms (7.23) and (7.25) may each be bounded by

$$\frac{C}{\sqrt{M}}\sqrt{rH(f|\nu_{\Lambda,r})}$$

in the following way. We again use Proposition 4.10 to get that

$$\nu_{\Lambda_k,R_k}[c_x(\eta_x)] - \mu_{\varphi_k}[c_x(\eta_x)] \le C\sqrt{1+\frac{R_k}{l}},$$

and hence we require $l = |\Lambda_k|$ to be sufficiently large. We begin with (7.25). Here we have that

$$\nu_{\Lambda,r}\left[f\cdot\sum_{k=1}^{m}\left\{\nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k}c_x(\eta_x)] - \mu_{\varphi_k}[\sum_{x\in\Lambda_k}c_x(\eta_x)]\right\}\right] \le C\nu_{\Lambda,r}\left[f\cdot\sum_{k=1}^{m}\sqrt{1+\frac{R_k}{l}}\right].$$

We proceed to bound this as

$$\begin{aligned}\nu_{\Lambda,r}\left[f\cdot\sum_{k=1}^{m}\sqrt{1+\frac{R_k}{l}}\right] &\le Cm\sqrt{1+\rho}\\ &\le Cml\sqrt{1+\rho}\sqrt{\frac{\rho}{Mr}}\sqrt{H(f|\mu_{N,r})}\\ &\le \frac{C}{\sqrt{M}}\sqrt{rH(f|\mu_{N,r})},\end{aligned}$$

where we used the fact that under case 1(c) we have $1 \le l\sqrt{\frac{\rho}{Mr}}\sqrt{H(f|\mu_{N,r})}$. We handle (7.23) in exactly the same way.

It remains to place bounds on (7.22) and (7.24). In fact, the bounds obtained for (7.22) will imply the necessary bounds for (7.24). We re-introduce the notation F_k defined in (7.11) thus obtaining

$$\nu_{\Lambda,r}\left[f\cdot\sum_{k=1}^{m}\left\{\mu_{\varphi_k}[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)] - \mu_{\Lambda,\varphi}[\sum_{x\in\Lambda_k}\tilde{c}_x(\eta_x)]\right\}\right] = \nu_{\Lambda,r}[f\cdot\sum_{k=1}^{m}lF_k] \tag{7.26}$$

Recall that

$$Z_k = \frac{1}{l}\sum_{x\in\Lambda_k}\frac{\eta_x-\rho_x}{\sigma_k(\varphi)},$$

and define a function G

$$G(z) = \begin{cases} C(B)z^2 & \text{for } |z|\le B\\ C(B)B^2 + 2BC(B)(|z|-B) & \text{for } |z|>B.\end{cases} \tag{7.27}$$

In the definition of G we choose the constant B so that $|F_k| \leq \sigma_k G(Z_k)$. We use this bound in (7.26) along with the second part of Proposition 4.4 to obtain

$$\begin{aligned} \nu_{\Lambda,r}[f; \sum_{k=1}^{m} lF_k] &\leq l\sigma \sum_{k=1}^{m} \nu_{\Lambda,r}[f \cdot G(Z_k)] \\ &\leq l\sigma \sum_{k=1}^{m} \{\nu_{\Lambda,r}[f; G(Z_k)] + \nu_{\Lambda,r}[G(Z_k)]\} \end{aligned} \tag{7.28}$$

We next introduce the fields $\mathcal{F}_k = \sigma\{\eta_x; x \in \Lambda_k \cup \Lambda_{k+1}\}$. For l sufficiently large we may use Propositions 4.10 and 4.4 to get the following:

$$G_k \equiv \nu_{\Lambda,r}[G(Z_k)|\mathcal{F}_k] \leq C(B)\frac{R_k + R_{k+1}}{l^2\sigma^2(\varphi)}. \tag{7.29}$$

These calculations allow us to bound the right hand side of (7.28):

$$\begin{aligned} l\sigma \sum_{k=1}^{m} \nu_{\Lambda,r}[G(Z_k)] &\leq \frac{Cr}{l\sigma} \\ &\leq C\sqrt{\frac{r}{M} H_{N,r}(f)} \end{aligned} \tag{7.30}$$

where we have used (7.21) in the last line. It remains to estimate the left hand side of (7.28).

$$l\sigma \sum_{k=1}^{m} \nu_{\Lambda,r}[f; G(Z_k)] = l\sigma \sum_{k=1}^{m} \nu_{\Lambda,r}[f; G(Z_k)] \tag{7.31}$$

$$+ l\sigma \sum_{k=1}^{m} \nu_{\Lambda,r}[\nu_{\Lambda,r}[f; G(Z_k)|\mathcal{F}_k]] \tag{7.32}$$

By an identical argument to that for (7.30) and the fact that $\nu_{\Lambda,r}[f] = 1$ we bound (7.32) by $C\sqrt{\frac{r}{M} H_{N,r}(f)}$. It remains to study (7.31). We use the notation $\nu_k[\cdot]$ for $\nu_{\Lambda,r}[\cdot|\mathcal{F}_k]$.

$$\begin{aligned} \nu_k[f; G(Z_k)] &\leq \left\{\nu_k\left[\left(\sqrt{f} - \nu_k[\sqrt{f}]\right)^2 \Big|G(Z_k) - G_k\Big|\right]\right\}^{1/2} \\ &\quad \times \left\{\nu_k\left[\left(\sqrt{f} + \nu_k[\sqrt{f}]\right)^2 \Big|G(Z_k) - G_k\Big|\right]\right\}^{1/2} \\ &= \sqrt{V_1 V_2}. \end{aligned} \tag{7.33}$$

Arguing as in (7.29), but using this time Lipshitz bounds on G, we have that

$$\nu_k[G_k] \leq C'(B)\frac{\sqrt{R_k + R_{k+1}}}{l\sigma},$$

for some constant $C'(B)$. Hence

$$V_1 \;\le\; C'(B)\nu_k\left[(\sqrt{f}-\nu_k[\sqrt{f}])^2\left|\frac{R_k-l\rho_k}{l\sigma}+\frac{\sqrt{R_k+R_{k+1}}}{l\sigma}\right|\right]$$

(where $C'(B)$ may be a new constant) which by the entropy inequality may be bounded by (writing $C=C'$)

$$\frac{C}{l\sigma}\nu_k\left[(\sqrt{f}-\nu_k[\sqrt{f}])^2\right]\left(\frac{1}{s}\log\nu_k[e^{s|R_k-l\rho_k|}]+\sqrt{R_k+R_{k+1}}\right)$$
$$+\;\frac{C}{sl\sigma}H((\sqrt{f}-\nu_k[\sqrt{f}])^2|\nu_k).$$

Notice that the above inequality holds because $\nu_k\left[(\sqrt{f}-\nu_k[\sqrt{f}])^2\right]$ is not equal to one. Using Lemma 7.6 in the above this is smaller than

$$V_1 \le \frac{C}{l\sigma}\nu_k[\sqrt{f};\sqrt{f}]\left(\frac{1}{s}+s(R_k+R_{k+1})+\sqrt{R_k+R_{k+1}}\right)$$
$$+\;\frac{C}{sl\sigma}H((\sqrt{f}-\nu_k[\sqrt{f}])^2|\nu_k),$$

where the constant C in front may depend on M. Optimizing over s we next get

$$V_1 \;\le\; \frac{C}{l\sigma}\nu_k[\sqrt{f};\sqrt{f}]\left(\sqrt{R_k+R_{k+1}}(1+\sqrt{\frac{H((\sqrt{f}-\nu_k[\sqrt{f}])^2|\nu_k)}{\nu_k[\sqrt{f};\sqrt{f}]}})\right)$$

Let $D_{k,k+1}(\sqrt{f})$ denote the Dirichlet form of the process defined over $\Lambda_k\cup\Lambda_{k+1}$. We next apply the logarithmic Sobolev inequality in the line above, with constant $\kappa=\kappa(2l)$

$$\frac{C\sqrt{R_k+R_{k+1}}}{l\sigma}\nu_k[\sqrt{f};\sqrt{f}]+\frac{C\sqrt{R_k+R_{k+1}}}{l\sigma}\sqrt{\kappa(2l)\nu_k[\sqrt{f};\sqrt{f}]D_{k,k+1}(\sqrt{f})}.$$

We repeat the argument for V_2 to obtain

$$V_2 \;\le\; \frac{C\sqrt{R_k+R_{k+1}}}{l\sigma}\nu_k[f]+\frac{C\sqrt{R_k+R_{k+1}}}{l\sigma}\sqrt{\kappa(2l)\nu_k[f]D_{k,k+1}(\sqrt{f})}.$$

We now use the spectral gap result $\nu_k[\sqrt{f};\sqrt{f}]\le Cl^2D_k(\sqrt{f})$ and $\nu_k[\sqrt{f};\sqrt{f}]\le\nu_k[f]$ in the above bounds to obtain for some constant C depending on l

$$\nu_{\Lambda,r}[\nu_k[f;G(Z_k)]]\le\frac{C}{\sigma}\sqrt{\nu_{\Lambda,r}[(R_k+R_{k+1})f]\nu_{\Lambda,r}[D_{k,k+1}(\sqrt{f})]}.$$

We use this to compute the quantity of interest

$$\begin{aligned} l\sigma \sum_{k=1}^{m} \nu_{\Lambda,r}[f;G_k] &= l\sigma \sum_{k=1}^{m} \nu_{\Lambda,r}[\nu_k[f;G_k]] \\ &\leq l\sigma \sum_{k=1}^{m} \frac{C}{\sigma} \sqrt{\nu_{\Lambda,r}[(R_k+R_{k+1})f]\nu_{\Lambda,r}[D_k(\sqrt{f})]} \\ &\leq Cl \sum_{k=1}^{m} \sqrt{\nu_{\Lambda,r}[R_k f]\nu_{\Lambda,r}[D_k(\sqrt{f})]} \\ &\leq Clm\sqrt{r\nu_{\Lambda,r}[D_k(\sqrt{f})]} \\ &\leq CN\sqrt{rD_{\Lambda,r}(\sqrt{f})} \end{aligned}$$

We may now combine the above line together with the bounds obtained for (7.23),(7.24) and (7.25) to obtain that

$$\nu_{\Lambda,r}[f;\sum_{k=1}^{m} \nu_{\Lambda_k,R_k}[\sum_{x\in\Lambda_k} c_x(\eta_x)]]^2 \leq r\left(\frac{C}{M}H_{N,r}(f) + C(M)N^2 D_{\Lambda,r}(\sqrt{f})\right)$$

We combine the results of Case 1, (a) through (c), and choose $\epsilon = \frac{C}{M}$ to obtain Proposition 7.4 for any large density. Notice that in the above work although we need to choose l sufficiently large for certain bounds to hold, once we do so, it remains fixed. We show next the necessary bounds for small density with appropriate choice of cutoff ρ_0.

7.1.2. *Case 2. small density: $\rho \leq \rho_0$.*

Lemma 7.8. *For every $\epsilon > 0$ there exists a ρ_0 and a constant $C = C(\epsilon) > 0$ and an N_0 so that for $\frac{r}{N} \leq \rho_0$ and $N \geq N_0$*

$$\nu_{\Lambda,r}[f;\sum_{x\in\Lambda} c_x(\eta_x)]^2 \leq r\nu_{\Lambda,r}[f](C\nu_{\Lambda,r}[f] + \epsilon H(f|\nu_{\Lambda,r})).$$

Proof. We assume $\nu_{\Lambda,r}[f] = 1$, again without loss of generality. By the entropy inequality

$$\nu_{\Lambda,r}[f;\sum_{x\in\Lambda} c_x(\eta_x)] \leq \frac{1}{t}\log \nu_{\Lambda,r}[e^{t\sum_{x\in\Lambda}(c_x(\eta_x)-\nu_{\Lambda,r}[c_x(\eta_x)])}] + \frac{1}{t}H(f|\nu_{\Lambda,r}) \tag{7.34}$$

Now,

$$\nu_{\Lambda,r}[e^{t\sum_{x\in\Lambda}(c_x(\eta_x)-\nu_{\Lambda,r}[c_x(\eta_x)])}] \leq Ce^{Ct\sqrt{r}}\Pi_{x\in\Lambda}\mu_{\varphi(\frac{r}{N})}[e^{t(c_x(\eta_x)-\frac{\varphi}{\rho_x}\eta_x)}]$$

by the Cauchy-Schwarz inequality, and Propositions 4.9 and 4.11. We next handle the term $\mu_{\varphi(\frac{r}{N})}[e^{t(c_x(\eta_x)-\frac{\varphi}{\rho_x}\eta_x)}]$. We have previously obtained bounds in Lemma 7.5,

but these are not sufficient here. Fix a constant M and for t in $[0, M]$ and φ in $[0, 1]$ define

$$F(t, \varphi) = \mu_\varphi[e^{t(c_x(\eta_x) - \frac{\varphi}{\rho_x}\eta_x)}]. \tag{7.35}$$

We notice a few things about the function F:

- $F(0, \varphi) = 1$,
- $\partial_t F(0, \varphi) = 0$,
- $\partial_t^2 F(0, \varphi) = \mu_\varphi[(c_x(\eta_x) - \frac{\varphi}{\rho_x}\eta_x)^2]$.

We wish to bound these derivatives and integrate to obtain an appropriate bound on the function F. Because we are in the setting of bounded densities and bounded t the function F as well as its derivatives are well behaved. We also have the following:

- $\partial_t \partial_\varphi^k F(0, 0) = 0$ for all k,
- $\partial_\varphi \partial_t^2 F(0, 0) = 0$.

By the above, there exists a constant $C(M)$, possibly depending on M, such that

$$F(t, \varphi) \leq 1 + C(M)\varphi^2 t^2 \leq e^{C(M)\varphi^2 t^2} \tag{7.36}$$

for t in $[0, M]$ and $\varphi \leq 1$. We next replace φ in the above by $C\rho$. Inserting these bounds into the entropy inequality (7.34) we get

$$\nu_{\Lambda,r}[f; \sum_{x\in\Lambda} c_x(\eta_x)]^2 \leq \frac{C}{t^2} + Cr + C(M)r^2\rho^2 t^2 + \frac{1}{t^2} H^2(f|\nu_{\Lambda,r}),$$

for all ρ such that $\varphi(\rho) \leq 1$. We again wish to optimize over t. As before, choose

$$t^2 = \frac{1 \vee [MH(f|\nu_{\Lambda,r})]}{r}.$$

As long as this $t \leq M$ we may plug it into the above bound to get

$$\nu_{\Lambda,r}[f; \sum_{x\in\Lambda} c_x(\eta_x)]^2 \leq 2Cr + C(M)r\rho^2(1 \vee [MH(f|\nu_{\Lambda,r})]) + \frac{r}{M} H(f|\nu_{\Lambda,r}).$$

Otherwise, because $\sum_{x\in\Lambda} c_x(\eta_x) \leq r$, and $t > M$ implies that $rM < H$, we have the easier bound

$$\nu_{\Lambda,r}[f; \sum_{x\in\Lambda} c_x(\eta_x)]^2 \leq Cr^2 \leq \frac{Cr}{M} H(f|\nu_{\Lambda,r}).$$

We now choose $\epsilon = \frac{C}{M}$ and then ρ_0 small enough so that the result follows. □

7.2. **Proof of Proposition 7.2.**

Proof. In what follows, assume $\varphi = \varphi(\rho)$ with $\rho = \frac{r}{N}$. We will prove instead the inequality

$$\nu_{\Lambda,r}[f; \sum_{x\in\Lambda} \varphi h_x(\eta_x)]^2 \leq r\nu_{\Lambda,r}[f] \left[C\nu_{\Lambda,r}[f] + CN^2 D_{\Lambda,r}(\sqrt{f}) + \epsilon H(f|\nu_{\Lambda,r}) \right],$$

from which the desired result follows. Because c_x and φh_x are of the same order, we may follow the proof of Proposition 7.1 with only mild modifications. We first consider the case of ρ bounded below. All of the arguments go through as before, until (7.8), where we need to know that

$$|\nu_{\Lambda,r}[\varphi h_x(\eta_x)] - \mu_\varphi[\varphi h_x(\eta_x)]| \leq \frac{C}{N}\sqrt{1+\rho}.$$

This is a consequence of Proposition 4.11 as in the case of c_x as long as we can show that

$$\mu_{\Lambda,\varphi}[h_x; h_x] \leq \frac{C}{\rho}.$$

Because of Lemma 5.2 we have

$$\mu_{\Lambda,\varphi}[h_x; h_x] \quad \leq \quad B\mu_{\Lambda,\varphi}[c_x(\eta_x)(h_x(\eta_x - 1) - h_x(\eta_x))^2],$$

where B holds for all x. This last quantity is less than

$$\begin{aligned} & B\mu_{\Lambda,\varphi}\left[\frac{1}{c_x(\eta_x)c_x^2(\eta_x+1)}(\eta_x(c_x(\eta_x+1) - c_x(\eta_x)))^2\right] \\ \leq \quad & CB\mu_{\Lambda,\varphi}\left[\frac{1}{c_x(\eta_x+1)}\right] \\ \leq \quad & \frac{CB}{\rho}, \end{aligned}$$

for some positive constant C. To continue with the argument we need to also specify how to handle the Taylor approximation arguments involving the functions $\tilde{c}_x$ and F_k (cf. (7.3) and (7.11)). Here we use

$$\varphi\tilde{h}_x = \varphi h_x - \varphi\gamma_k'(\rho_k)\eta_x,$$

for all x in Λ_k where $\gamma_k(x) = \frac{x}{\varphi_k(x)}$. This implies that the new version of F_k becomes

$$\begin{aligned} F_k \quad &= \quad \varphi\left[\frac{\frac{R_k}{l}}{\varphi_k(\frac{R_k}{l})} - \frac{\rho_k}{\varphi(\rho)}\right] - \varphi\gamma_k'(\rho_k)\left[\frac{R_k}{l} - \rho_k\right] \\ &= \quad \varphi\left[\frac{\frac{R_k}{l}}{\varphi_k(\frac{R_k}{l})} - \frac{\rho_k}{\varphi_k(\rho_k)}\right] - \varphi\gamma_k'(\rho_k)\left[\frac{R_k}{l} - \rho_k\right]. \end{aligned}$$

So that the argument of the previous section goes through we need to know two things: we need δ_k to be uniformly bounded and we need to be able to choose a constant B so that $|F_k| \leq \sigma_k G(Z_k)$ with G and Z defined in (7.27) and (7.12).

By direct calculation we have

$$\gamma_k'(x) = \frac{1}{\varphi_k(x)}\left\{1 - \frac{x}{\sigma_k^2(x)}\right\}, \tag{7.37}$$

which implies that $\varphi\gamma_k'(\rho_k)$ is uniformly bounded by Proposition 4.4.

We next need to show that $\varphi\gamma_k$ is Lipschitz and that we can choose a B so that $\gamma_k''(x)$ is bounded when $|x-\rho_k| \le B\sigma_k$. We first show that

$$|\gamma_k(x) - \gamma_k(\rho_k)| \le \frac{C}{\varphi}\, |x-\rho_k|.$$

Because γ_k is uniformly bounded (in k) the inequality is immediate for $|x-\rho_k| > \frac{\rho_k}{2}$. Also, by (7.37), we know that $\gamma_k'(x)$ is bounded for x away from zero, and this gives us the necessary bound for $|x-\rho_k|$ small. Lastly, we need to show that the second derivative of γ_k is well behaved when $|x-\rho_k| \le B\sigma_k$ for some choice of B, so that we may obtain the tighter bounds for smaller values of Z_k. We calculate directly

$$\gamma''(x) = -2\frac{\varphi_k'(x)}{\varphi_k^2(x)} + 2x\frac{\varphi_k'(x)^2}{\varphi_k^3(x)} - x\frac{\varphi_k''(x)}{\varphi_k^2(x)}.$$

Careful examination of the above reveals that this does indeed remain bounded for x away from zero, and hence we have the necessary bounds as long as we consider ρ away from zero and B sufficiently small. The rest of the proof of Proposition 7.2 follows as before for the case of ρ bounded below.

For the case of ρ small (bounded above), we need to specify the version of

$$c_x(\eta_x) - \frac{\varphi}{\rho_x}\eta_x$$

required in the definition of (7.35). We use in this case

$$\varphi h_x(\eta_x) - \rho_x.$$

This function defines the new $F(t,\varphi) = \mu_\varphi[e^{t(\varphi h_x(\eta_x)-\rho_x)}]$ in lieu of (7.35) from before. It is straightforward to check that this new function satisfies all of the required properties so that we obtain the appropriate bounds (7.36). The rest of the proof goes through without further changes.

□

8. Proof of Spectral Gap for Inhomogeneous Zero Range.

This section is dedicated to the proof of the Theorem 2.3. The method is the same as that used in [LSV], while carefully making sure all necessary bounds hold uniformly in the sites. We will frequently make use of the spectral gap results obtained in section 5 for certain birth and death processes. We present the proof for a general dimension d, in order to highlight the changes needed to extend the proof of the logarithmic Sobolev inequality to higher dimensions.

8.1. **Outline of Proof.** Let $\omega(N,r)$ be the smallest constant such that

$$\nu_{\Lambda,r'}[f;f] \le \omega(N,r) D_{\Lambda,r'}(f)$$

for all $|\Lambda| \le N^d$ and all $r' \le r$. Also, let $\omega(N) = \sup_k \omega(N,k)$.

The general approach here is quite similar to the one in the logarithmic Sobolev inequality: using induction we establish two recursive equations for ω. The first equation allows us to establish that the constant is free of the number of particles, while the second, valid only for sufficiently large N, gives the N^2 order. The difference with the approach of the logarithmic Sobolev inequality is that the induction increment *adds* one to the side length of the cube Λ, and does not *double* it.

Assume then that Λ is a set of size $|\Lambda| = N^d$, and write $\Lambda = \Lambda_0 \cup \Lambda_1$, where $|\Lambda_1| = (N-1)^d$ and Λ_1 is still a cube containing one of the corner points. That is, if $d = 1$ then $\Lambda_1 = \{1, \ldots, N-1\}$, if $d = 2$ then $\Lambda_1 = \{(z_1, z_2); z_i = 1, \ldots, N-1\}$, and so forth. Denote by $R_0 = R_0(\eta)$ the random variable counting the number of particles in $\Lambda_0 = \Lambda \setminus \Lambda_1$. Also, enumerate the sites of Λ_0 so that $\eta_{\Lambda_0} = \{\eta_{z_k}\}_{k=1}^{N^*}$, where $N^* = N^d - (N-1)^d$. For simpilicity, we denote z_k simply as k. Lastly, let $\mathcal{F}_k = \sigma\{\eta_1, \ldots, \eta_k\}$ denote the σ-algebra generated by the first k elements of Λ_0. Thus, $\mathcal{F}_k$ forms an increasing filtration, and denoting $\nu_{\Lambda,r}[f|\mathcal{F}_k]$ by f_k, we may write

$$\nu_{\Lambda,r}[f;f] = \nu_{\Lambda,r}[\nu_{\Lambda_1,r-R_0}[f;f]] + \sum_{k=0}^{N^*} \nu_{\Lambda,r}[(f_{k+1} - f_k)^2]. \tag{8.1}$$

Here, $\mathcal{F}_0$ denotes the trivial σ-algebra. By the induction hypothesis, we may bound the first term above by

$$\omega(N-1)\nu_{\Lambda,r}[D_{\Lambda_1,r-R_0}(f)] \leq \omega(N-1)D_{\Lambda,r}(f). \tag{8.2}$$

To bound the second term we write

$$\begin{aligned} \nu_{\Lambda,r}[(f_{k+1} - f_k)^2] &= \nu_{\Lambda,r}[\nu_{\Omega_k,R_k}[(f_{k+1} - f_k)^2]] \\ &= \nu_{\Lambda,r}[\nu_{\Omega_k,R_k}[f_{k+1}; f_{k+1}]] \end{aligned}$$

where $\Omega_k = \Lambda \setminus \{z_m \in \Lambda_0\}_{m=1}^k$ and R_k is the number of particles there. Restricting consideration to the measure ν_{Ω_k,R_k} we think of f_{k+1}, a function of $\{\eta_m\}_{m=1}^{k+1}$ as a function only of η_{k+1}, imagining the remaining sites to be fixed. We thus write f_{k+1}, with a slight abuse of notation, as $\phi_k(\eta_{k+1})$. We obtain

$$\begin{aligned} \nu_{\Omega_k,R_k}[f_{k+1}, f_{k+1}] &= \nu_{\Omega_k,R_k}[\phi_k, \phi_k] \\ &\leq B_1\ \nu_{\Omega_k,R_k}[c_k(\eta_{k+1})\{\phi_k(\eta_{k+1} - 1) - \phi_k(\eta_{k+1})\}^2], \end{aligned} \tag{8.3}$$

by Lemma 5.1, where the constant B_1 does not depend on the location of the site $k+1$. We next write (8.3) as

$$B_1 \sum_{m=0}^{R_k - 1} \nu_{\Omega_k,r_k}(\eta_{k+1} = m+1)c_{k+1}(m+1)\{\phi_k(m+1) - \phi_k(m)\}^2. \tag{8.4}$$

Using a calculation similar to the more general one of Proposition 6.1 (indeed, this is just a special case of that result) we obtain that

$$\begin{aligned}\phi_k(m+1)-\phi_k(m) &= \nu_{\Omega_k,R_k}[f|\eta_{k+1}=m+1]-\nu_{\Omega_k,R_k}[f|\eta_{k+1}=m]\\ &= \frac{1}{\nu_{\Omega_k,R_k}(\eta_{k+1}=m+1)c_k(m+1)}\left\{A_1+A_2\right\},\end{aligned}$$

where

$$\begin{aligned}A_1(k,\Lambda,r,f) &= AV_{y\in\Omega_{k+1}}\nu_{\Omega_k,R_k}[c_y(\eta_y)\nabla_{y,k+1}f\ \mathbb{I}(\eta_{k+1}=m)],\\ A_2(k,\Lambda,r,f) &= \nu_{\Omega_k,R_k}[\eta_{k+1}=m]\nu_{\Omega_k,R_k}[f;AV_{y\in\Lambda_{k+1}}c_y(\eta_y)|\eta_{k+1}=m].\end{aligned}$$

This means that (8.3) is bounded above by (a constant, $2B_1$, times) the sum of T_1 and T_2, where

$$T_1=\sum_{m=0}^{R_k-1}\frac{\left\{AV_{y\in\Omega_{k+1}}\nu_{\Omega_k,R_k}[c_y(\eta_y)\nabla_{y,k+1}f\ \mathbb{I}(\eta_{k+1}=m)]\right\}^2}{\nu_{\Omega_k,R_k}(\eta_{k+1}=m+1)c_k(m+1)},\quad\text{and}$$

$$T_2=\sum_{m=0}^{R_k-1}\frac{\nu_{\Omega_k,R_k}(\eta_{k+1}=m)}{\nu_{\Omega_{k+1},R_k-m}[c_k(\eta_k)]}\left\{\nu_{\Omega_k,R_k}[f;AV_{y\in\Lambda_{k+1}}c_y(\eta_y)|\eta_{k+1}=m]\right\}^2,$$

and we have used the relation

$$\nu_{\Lambda\setminus\{z\},r-\eta_z}[c_y(\eta_y)]\nu_{\Lambda,r}(\eta_z=m)=\nu_{\Lambda,r}(\eta_z=m+1)c_z(m+1),$$

for any $z,y\in\Lambda$, in the latter. The next steps establish bounds on these terms.

Proposition 8.1. *There exists a finite constant C such that*

$$\sum_{k=1}^{N^*}\nu_{\Lambda,r}[T_1(k,\Lambda,r,f)]\le CND_{\Lambda,r}(f).$$

This is a universal bound on T_1. We also need to establish both a weak and a strong version of bounds on T_2, to be used in the recursive equations.

Proposition 8.2. *There exists a finite constant C such that*

$$T_2(k,\Lambda,r,f)\le C\omega(N-1)D_{\Omega_k,R_k}(f).$$

Using local limit theorems, this may be tightened for sufficiently large values of N.

Proposition 8.3. *For all $\epsilon>0$, there exist finite constants $n_0(\epsilon)$ and $C(\epsilon)$ such that*

$$T_2(k,\Lambda,r,f)\le C(\epsilon)N^{-d}D_{\Omega_k,R_k}(f)+\epsilon N^{-d}\nu_{\Lambda_K,R_k}[f;f]$$

for all $n\ge n_0$.

We may now combine these propositions to prove the result. First, Propositions 8.1 and 8.2 applied in (8.1) together with (8.2) give

$$\begin{aligned}
\nu_{\Lambda,r}[f;f] &\leq \omega(N-1)D_{\Lambda,r}(f) + \sum_{k=0}^{N^*} \nu_{\Lambda,r}[\nu_{\Omega_k,R_k}[f_{k+1};f_{k+1}]] \\
&\leq \omega(N-1)D_{\Lambda,r}(f) + CND_{\Lambda,r}(f) + CN^{d-1}\sum_{k=0}^{N^*}\omega(N-1)\nu_{\Lambda,r}[D_{\Omega_k,R_k}(f)] \\
&\leq \left\{\left(1+CN^{d-1}\right)\omega(N-1) + CN\right\} D_{\Lambda,r}(f), \qquad (8.5)
\end{aligned}$$

since $\nu_{\Lambda,r}[D_{\Omega_k,R_k}(f)] \leq D_{\Lambda,r}(f)$ and $N^* \leq CN^{d-1}$ for some constant C. Tightening these bounds using Proposition 8.3 we have for any ϵ

$$\nu_{\Lambda,r}[f;f] \leq \{CN + C(\epsilon) + \omega(N-1)\} D_{\Lambda,r}(f) + B_1\epsilon N^{-d}\nu_{\Lambda,r}[f;f] \qquad (8.6)$$

for sufficiently large N.

Notice that the initial induction step, $\omega(2) < \infty$, is established in Lemma 5.1, because when $|\Lambda| = 2$, then $f(\eta) = f(\eta_1, r-\eta_1) = \phi(\eta_1)$. From (8.5) we have that there is some constant, $C(N)$, independent of the number of particles, such that

$$\omega(N) \leq C(N)\omega(N-1) + \frac{N}{2}.$$

This recursive equation implies that $\omega(N)$ is finite for every N.

Similarly using the tighter bounds from (8.6) we obtain also that for all ϵ and sufficiently large values of N

$$\omega(N) \leq (1-\epsilon/N^d)^{-1}\left\{B_1C(\epsilon) + \omega(N-1) + CN\right\},$$

which implies the required quadratic growth. Thus the two recursive formulae above, along with the initial induction step, establish Theorem 2.3.

8.2. **Proof of Proposition 8.1.** From

$$\nu_{\Omega,m}[c_z(\eta_z)\mathbb{I}(\eta_y = k)] = \nu_{\Omega,m}(\eta_y = k+1)c_y(k+1),$$

it follows by the Schwarz inequality that

$$
\begin{aligned}
&\sum_{k=1}^{N^*} \nu_{\Lambda,r}[T_1(k,\Lambda,r,f)] \\
&= \sum_{k=1}^{N^*} \nu_{\Lambda,r}\left[\sum_{m=0}^{R_k-1} \frac{\left\{AV_{y\in\Omega_{k+1}}\nu_{\Omega_k,R_k}[c_y(\eta_y)\nabla_{y,k+1}f\ \mathbb{I}(\eta_{k+1}=m)]\right\}^2}{\nu_{\Omega_k,R_k}(\eta_{k+1}=m+1)c_k(m+1)}\right] \\
&\le \sum_{k=1}^{N^*} \nu_{\Lambda,r}\left[\nu_{\Omega_k,R_k}[AV_{y\in\Omega_{k+1}}c_y(\eta_y)\{\nabla_{y,k+1}f\}^2]\right] \\
&= \frac{C}{N^d}\sum_{k=1}^{N^*}\sum_{y\in\Omega_{k+1}} \nu_{\Lambda,r}[c_y(\eta_y)\{\nabla_{y,k+1}f\}^2]. \qquad (8.7)
\end{aligned}
$$

We next bound $\{\nabla_{y,k+1}f\}^2$ by

$$
CN\sum_{\{e_z\}}\{f(\eta^{x,e_{z+1}})-f(\eta^{x,e_z})\}^2,
$$

where the sum is over $\{e_z\}$: sites which form a path from y to $k+1$. We pick these paths in a particular way. We number the directions from 1 to d. The path from a site y to a site $k+1$ is a path such that we move maximally in the first direction, then maximally in the second direction, etc.. For example, in $d=2$ to join a site $x=\{x_1,x_2\}$ with a site $y=\{y_1,y_2\}$ such that $x_1>y_1$ and $x_2<y_2$ we choose the path such that there exists and m so that $e_1=\{x_1,x_2\}, e_2=\{x_1-1,x_2\},\ldots,\ e_m=\{y_1,x_2\}$ and $e_{m+1}=\{y_1,x_2+1\}, e_{m+2}=\{y_1,x_2+2\},\ldots,\ e_{m^*}=\{y_1,y_2\}$. Here $m^*=|x_1-y_1|+|x_2-y_2|+1$. Using this decomposition we have that

$$
\begin{aligned}
&\nu_{\Lambda,r}\left[c_y(\eta_y)\{\nabla_{y,k+1}f\}^2\right] \\
&\quad\le CN\sum_{\{e_z\}}\nu_{\Lambda,r}[c_y(\eta_y)\{f(\eta^{x,e_{z+1}})-f(\eta^{x,e_z})\}^2] \\
&\quad= CN\sum_{\{e_z\}}\nu_{\Lambda,r}[c_{e_z}(\eta_{e_z})\{\nabla_{e_z,e_{z+1}}f\}^2].
\end{aligned}
$$

By changing the order of summation, we conclude that (8.7) is bounded above by

$$
\begin{aligned}
&\frac{CN}{N^d}\sum_{k=1}^{N^*}\sum_{y\in\Omega_{k+1}}\sum_{\{e_z\}}\nu_{\Lambda,r}[c_{e_z}(\eta_{e_z})\{\nabla_{e_z,e_{z+1}}f\}^2] \\
&\le \frac{CN}{N^d}\sum_{w\sim v\in\Lambda}\nu_{\Lambda,r}[c_{e_z}(\eta_{e_z})\{\nabla_{e_z,e_{z+1}}f\}^2]\ \sum_{k,x}1\ ,
\end{aligned}
$$

where the last sum is taken over k, x all sites k and x such that both sites w and v are in the path $\{e_z\}$ from x to k. Because of our construction, this last quantity is bounded by a constant times N^d. This concludes the proof.

8.3. **Proof of Proposition 8.2.** By the Schwarz inequality

$$\begin{aligned}&\left\{\nu_{\Omega_k,R_k}[f; AV_{y\in\Omega_{k+1}}c_y(\eta_y)|\eta_{k+1}=m]\right\}^2 \\ &\qquad\leq \quad \nu_{\Omega_{k+1},R_k-m}[f;f]\nu_{\Omega_{k+1},R_k-m}[AV_{y\in\Omega_{k+1}}c_y(\eta_y); AV_{y\in\Omega_{k+1}}c_y(\eta_y)]. \end{aligned} \tag{8.8}$$

For any set Ω and R, and fixed $z\in\Omega$, we apply Lemma 5.1

$$\begin{aligned}\nu_{\Omega,R}[AV_{y\in\Omega}c_y(\eta_y); AV_{y\in\Omega}c_y(\eta_y)] \quad &\leq \quad \nu_{\Omega,R}[c_z(\eta_z); c_z(\eta_z)] \\ &\leq \quad B_1\nu_{\Omega,R}[c_z(\eta_z)\{c_z(\eta_z-1)-c_z(\eta_z)\}^2].\end{aligned}$$

By assumption (LG) this last expression is smaller than $a_1^2\nu_{\Omega,R}[c_z(\eta_z)]$. Plugging this back into (8.8), we obtain

$$T_2 \leq \nu_{\Omega_k,R_k}[\nu_{\Omega_{k+1},R_k-\eta_{k+1}}[f;f]].$$

Applying the induction hypothesis to this last statement, we show

$$\begin{aligned}T_2 \quad &\leq \quad \nu_{\Omega_k,R_k}[\omega(N-1)D_{\Omega_{k+1}}(f)] \\ &\leq \quad \omega(N-1)D_{\Omega_k,R_k}(f),\end{aligned}$$

as desired.

8.4. **Proof of Proposition 8.3.** Set $\Omega^+ = \Omega\cup\{z\}$ for a fixed site z, where Ω is such that $|\Omega|\geq CN^d$. Fix another site $x\in\Omega$. Using this notation we re-write our goal in simpler form: for all $\epsilon>0$, there exist finite constants $N_0(\epsilon)$ and $C(\epsilon)$ such that

$$\begin{aligned}&\sum_{m=0}^{R-1}\frac{\nu_{\Omega^+,R}(\eta_z=m)}{\nu_{\Omega,R-m}[c_x(\eta_x)]}\left\{\nu_{\Omega,R-m}[f; AV_{y\in\Omega}c_y(\eta_y)]\right\}^2 \\ &\qquad\leq \quad C(\epsilon)N^{1-d}D_{\Omega^+,R}(f)+\epsilon N^{-d}\nu_{\Omega^+,R}[f;f]\end{aligned} \tag{8.9}$$

for all $N\geq N_0$.

The proof of this is split into two cases: that of small density and that of large density. To this end, let $\rho=\frac{R}{|\Omega^+|}$, and fix $\rho_0>0$.

Case 1. $\rho\leq\rho_0$. By the Schwarz inequality we have that

$$\nu_{\Omega,R-m}[f; AV_{y\in\Omega}c_y(\eta_y)]^2 \leq \nu_{\Omega,R-m}[f;f]\nu_{\Omega,R-m}[AV_{y\in\Omega}c_y(\eta_y); AV_{y\in\Omega}c_y(\eta_y)].$$

Using change of measure we may write

$$\begin{aligned}&\frac{1}{\nu_{\Omega,R-m}[c_z(\eta_z)]}\nu_{\Omega,R-m}[AV_{y\in\Omega}c_y(\eta_y); AV_{y\in\Omega}c_y(\eta_y)] \\ &= \nu_{\Omega,R-m-1}[c_z(\eta_z)]-\nu_{\Omega,R-m}[c_z(\eta_z)]+\frac{1}{|\Omega|}\nu_{\Omega,R-m-1}[c_z(\eta_z+1)-c_z(\eta_z)]\end{aligned} \tag{8.10}$$

We wish to bound the term in (8.10) by $\frac{\epsilon}{|\Omega|}$ for all sufficiently large $|\Omega|$. For $\rho \geq \frac{A}{|\Omega|}$ this follows from Proposition 4.12. Otherwise the number of particles is bounded and we have that the Poisson limit theorem holds. Here again we obtain the desired bounds from Lemma 4.8.

Case 2. $\rho > \rho_0$. This is the more involved case of the two, and it requires a "two-block" argument. That is, we write Ω as a union of smaller cubes $B_1 \cup \cdots \cup B_K$, where for simplicity we assume that each cube is exactly of size l^d. As in the previous section we shall pick l to be a fixed quantity, however, sufficiently large so that certain estimates hold. We write as R_j the number of particles on cube B_j, $j = 1, \ldots, K$. To simplify notation we also write $\tilde{R}$ instead of $R - m$. By the Schwarz inequality we write

$$\begin{aligned} \nu_{\Omega,\tilde{R}}[f; AV_{y\in\Omega} c_y(\eta_y)]^2 \quad &\leq \quad 2\nu_{\Omega,\tilde{R}}\left[f; AV_{y\in\Omega}\{c_y(\eta_y) - \nu_{B_j,R_j}[c_y(\eta_y)]\mathbb{I}(y \in B_j)\}\right]^2 \\ &+ \quad 2\nu_{\Omega,\tilde{R}}\left[f; AV_j \nu_{B_j,R_j}[c_y(\eta_y)]\right]^2 \end{aligned}$$

We first handle the first term on the right hand side. We may write this as

$$\begin{aligned} &\left| \frac{1}{|\Omega|} \sum_{j=1}^{K} |B_k| \ \nu_{\Omega,\tilde{R}}\left[\nu_{B_j,R_j}\left[f; AV_{y\in B_j} c_y(\eta_y)\right]\right] \right| \\ &\qquad \leq \quad \frac{\alpha}{2|\Omega|} \sum_k |B_k| \ \nu_{\Omega,\tilde{R}}\left[\nu_{B_j,R_j}\left[f; f\right]\right] \\ &\qquad\qquad + \frac{1}{2\alpha|\Omega|} \sum_{j=1}^{K} |B_k| \ \nu_{\Omega,\tilde{R}}\left[\nu_{B_j,R_j}\left[AV_{y\in B_j} c_y(\eta_y); AV_{y\in B_j} c_y(\eta_y)\right]\right], \end{aligned} \tag{8.11}$$

for any strictly positive α. By the induction assumption we have

$$\sum_k |B_k| \ \nu_{\Omega,\tilde{R}}\left[\nu_{B_j,R_j}\left[f; f\right]\right] \leq l^d \omega(l+1) D_{\Omega,\tilde{R}}(f).$$

On the other hand, by the Schwarz inequality we have for the second term

$$\frac{1}{|\Omega|} \sum_{j=1}^{K} |B_k| \nu_{\Omega,\tilde{R}}\left[\nu_{B_j,R_j}\left[AV_{y\in B_j} c_y(\eta_y); AV_{y\in B_j} c_y(\eta_y)\right]\right] \leq AV_{y\in\Omega} \nu_{\Omega,\tilde{R}}\left[c_y(\eta_y); c_y(\eta_y)\right],$$

which, by Proposition 4.10, is bounded above by $C\rho$, for some constant C, and sufficiently large $|\Omega|$. Plugging these bounds into (8.11) and optimising in α we obtain that

$$\begin{aligned} &2\nu_{\Omega,\tilde{R}}\left[f; AV_{y\in\Omega}\{c_y(\eta_y) - \nu_{B_j,R_j}[c_y(\eta_y)]\mathbb{I}(y \in B_j)\}\right]^2 \\ &\leq C l^d \omega(l+1) \frac{1}{|\Omega|} D_{\Omega,\tilde{R}}(f) \nu_{\Omega,\tilde{R}}[c_x(\eta_x)], \end{aligned}$$

where we have also used the fact that there exists a positive constant such that $\rho \leq C\nu_{\Omega,\tilde{R}}[c_x(\eta_x)]$.

We next turn our attention to the second term. Using a similar argument to that of Section 7 we write

$$
\begin{aligned}
&2\nu_{\Omega,\tilde{R}}\left[f; AV_j \nu_{B_j,R_j}[c_y(\eta_y)]\right]^2 \\
&= 2\nu_{\Omega,\tilde{R}}\left[f; \frac{1}{|\Omega|}\sum_{j=1}^{K}|B_j|\left\{\nu_{B_j,R_j}[c_y(\eta_y)] - \varphi(\tilde{\rho}) - \varphi_j'(\rho_j)\left\{AV_{y\in B_j}\eta_y - \rho_j\right\}\right\}\right]^2,
\end{aligned}
$$

where $\tilde{\rho} = \tilde{R}/|\Omega|$, $\rho_j = \mu_{\Omega,\varphi(\tilde{\rho})}[AV_{y\in B_j}\eta_y]$ and $\varphi_j(\rho_j) = \mu_{B_j,\varphi(\tilde{\rho})}[AV_{y\in B_j}c_y(\eta_y)]$. Notice that $\varphi(\tilde{\rho}) = \varphi_j(\rho_j)$ (however, also note that $\varphi'(\tilde{\rho})$ is not equal to $\varphi_j'(\rho_j)$). We let $m_j = AV_{y\in B_j}\eta_y = R_j/l$ and set

$$F_j(m_j) = \nu_{B_j,R_j}[c_y(\eta_y)] - \varphi(\tilde{\rho}) - \varphi_j'(\rho_j)\left\{m_j - \rho_j\right\}.$$

With this notation we bound the last line above using the Schwarz inequality by

$$2\nu_{\Omega,\tilde{R}}[f;f]\nu_{\Omega,\tilde{R}}\left[\left(\frac{1}{|\Omega|}\sum_{j=1}^{K}|B_j|F_j(m_j)\right)^2\right].$$

We then write

$$
\begin{aligned}
&\nu_{\Omega,\tilde{R}}\left[\left(\frac{1}{|\Omega|}\sum_{j=1}^{K}|B_j|F_j(m_j)\right)^2\right] \\
&\qquad = \frac{l^{2d}}{|\Omega|^2}\sum_{j=1}^{K}\nu_{\Omega,\tilde{R}}[F_j^2(m_j)] + \frac{l^{2d}}{|\Omega|^2}\sum_{j\neq i}\nu_{\Omega,\tilde{R}}[F_j(m_j)F_i(m_i)]. \qquad (8.12)
\end{aligned}
$$

We next use the second part of Proposition 4.11 to switch to the grand canonical measure. This will allow us to take advantage of the Taylor series expansion we have set up. We do not use Proposition 4.10 here as we will take advantage of the freedom of making l large. Hence, we require the condition $\tilde{\rho} > \rho_0$ to make the argument. Thus,

$$\nu_{\Omega,\tilde{R}}[F_j^2(m_j)] \;\; \leq \;\; \mu_{\varphi(\tilde{\rho})}[F_j^2(m_j)] + E_0(\rho_0)\frac{l^d}{|\Omega|}\{\mu_{\varphi(\tilde{\rho})}[F_j^4(m_j)]\}^{1/2},$$

and similarly

$$
\begin{aligned}
\nu_{\Omega,\tilde{R}}[F_j(m_j)F_i(m_i)] \;\; &\leq \;\; \mu_{\varphi(\tilde{\rho})}[F_j(m_i)]\mu_{\varphi(\tilde{\rho})}[F_i(m_i)] \\
&\qquad + \;\; E_0(\rho_0)\frac{l^d}{|\Omega|}\{\mu_{\varphi(\tilde{\rho})}[F_j^2(m_j)]\mu_{\varphi(\tilde{\rho})}[F_i^2(m_i)]\}^{1/2} \\
&= \;\; E_0(\rho_0)\frac{l^d}{|\Omega|}\{\mu_{\varphi(\tilde{\rho})}[F_j^2(m_j)]\mu_{\varphi(\tilde{\rho})}[F_i^2(m_i)]\}^{1/2}.
\end{aligned}
$$

Applying the first part of Corollary 4.11 to $\nu_{B_j,R_j}[c_y(\eta_y)]$, and using Propositions 4.4, 4.5, and Corollary 4.6 to bound the resulting moments, we obtain that

$$\begin{aligned}\mu_{\varphi(\tilde\rho)}[F_j^4] &\le \mu_{\varphi(\tilde\rho)}\left[\left\{\frac{C(\rho_0)}{l^d}\sqrt{1+\tilde\rho}+\tilde c(m_j-\rho_j)\right\}^4\right]\\ &\le C'(\rho_0)\frac{1}{l^{2d}}\tilde\rho^2,\end{aligned}$$

for l and $|\Omega|$ sufficiently large. For the quadratic term we have

$$\begin{aligned}2^{-1}\mu_{\varphi(\tilde\rho)}[F_j^2] &\le \mu_{\varphi(\tilde\rho)}\left[\nu_{B_j,R_j}\left[c_y(\eta_y)-\varphi(m_j)\right]^2\right]\\ &\quad+\mu_{\varphi(\tilde\rho)}\left[\left\{\varphi(m_j)-\varphi_k(\rho_k)-\varphi_k'(\rho_k)(m_k-\rho_k)\right\}^2\right]\end{aligned}$$

By Corollaries 4.11 and 4.6 we bound the first term above by $C\frac{1}{l^{2d}}(1+\ \tilde\rho)$, for some constant C. The second term may be bounded by $C\frac{1}{l^{2d}}\tilde\rho$ using (7.14) and (7.15) (noting that those particular arguments do not depend on the dimension). We now put all of the above work together to obtain the bound in (8.12)

$$\nu_{\Omega,\tilde R}\left[\left(\frac{1}{|\Omega|}\sum_{j=1}^{K}|B_j|F_j(m_j)\right)^2\right]\le C(\rho_0)\frac{1}{|\Omega|}\frac{1}{l^d}\tilde\rho. \tag{8.13}$$

We may now select $l=\epsilon^{-d}$. This completes the proof of (8.9).

Notice that in the above arguments it is only Proposition 8.1 which is sensitive to the geometry of the problem induced by change in dimension.

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